A generalized permutohedron is a polytope obtained by moving the facets (in parallel) without them going across vertices.

\[
\begin{align*}
\text{Example:} & \\
\text{Prop/Def. Generalized permutohedra are precisely the polytopes } & \\
\text{GP}(z) \text{ of the form } & \\
\sum_{i \in E} x_i = z(E) \quad (E = \{1, \ldots, n\}) \\
\sum_{i \in A} x_i & \leq z(A) \\
\text{for a submodular function } & \\
z : 2^E \to \mathbb{R}_+ \{0\} \\
\text{i.e., one such that } & \\
z(A) + z(B) \geq z(A \cup B) + z(A \cap B) \\
\text{for all } A, B \subseteq E
\end{align*}
\]
Ex 1 (Graphs)

Given a graph $G = (V, E)$, let

$Z_G : 2^V \rightarrow \mathbb{R}$

$Z_G(W) = \# \text{ of edges incident to a vertex in } W$

$G = \begin{array}{c}
\begin{array}{ccc}
1 & 3 & 4 \\
\downarrow & \downarrow & \downarrow \\
2 & & \\
\end{array}
\end{array}$

$Z(\emptyset) = 0$
$Z(\{1\}) = Z(\{2\}) = Z(\{3\}) = 3, \quad Z(\{4\}) = 1$
$Z(\{1,2\}) = 4, \quad Z(\{1,3\}) = 5, \ldots$
$Z(\{1,2,3\}) = 5, \ldots$
$Z(\{1,2,3,4\}) = 5$

Exercise: $Z_G$ is submodular

$\text{GP}(Z_G)$ is the graphical zonotope of $G$

Ex 2 (Matroids)

The matroid polytope of a matroid $M$ is:

$P_M = \text{conv} \{ e_B : B \text{ basis of } M \} \subseteq \mathbb{R}^E$

where $E = \{e_1, \ldots, e_r\}$, $e_8 = 001010 \ldots 010010$

$P_M = \begin{array}{c}
\begin{array}{c}
11000 \\
10100 \\
10010 \\
11100 \\
\end{array}
\end{array}$

$\text{Ex}$ $B = \{123, 124, 125, 134, 135\}$

$P_M = \begin{array}{c}
\begin{array}{c}
10100 \\
10100 \\
10100 \\
10100 \\
\end{array}
\end{array} \subseteq \mathbb{R}^6$

Ex 3 (Polytopes)

Given a poset $P$, let $Z_P : 2^P \rightarrow [0, \infty]$ be:

$Z_P(A) = \begin{cases}
0 & \text{if } A \text{ is a chain of } P \\
\infty & \text{otherwise}
\end{cases}$

Exercise: $Z_P$ is submodular

$\text{GP}(Z_P)$ is the poset polyhedron of $P$

The Hopf algebra $\text{GP}$:

Let $P = \text{GP}(Z)$ be a gen. perm. in $\mathbb{R}_E^E$, $P' = \text{GP}(Z')$

Then $P \times P'$ is a gen. perm. in $\mathbb{R}_{E \times E}$, with

$Z(A \cup A') = Z(A) + Z'(A')$

So $\text{GP}$ is closed under $\times$.

Now I want restriction, contraction.
Let \( P = GP(z) \) in \( \mathbb{P}^E \).
Let \( A \subset E \).
Consider the face 
\[
P_A = \{ p \in P : p \cdot e_A \text{ is maximal} \}.
\]

**Fact:**
\[
P_A = Q \times R \quad \text{for} \quad Q \subset R^A, \quad R \subset \mathbb{R}^{E-A}
\]

Let \( P_A = Q \) be the restriction, contraction
\[P/A = R\]

- \( P/A, P/A \) are gen. perm.

**Ex:**
\[
A = \{1, 4, 5\}
\]
\[
A = \{1, 4, 5\}
\]
\[
A = \{1, 4, 5\}
\]
\[
A = \{1, 4, 5\}
\]
\[
P_A = \begin{bmatrix} 110100 \\ 101010 \\ 101010 \end{bmatrix}
\]
\[
= P(A) \times P/A
\]
\[
S(P) = \sum \frac{(-1)^{\dim P - \dim Q}}{Q \text{ face}}
\]

**Theorem (Aguiar-Ardila):**
The antipode for generalized permutahedra is
\[
S(P) = \sum \frac{(-1)^{\dim P - \dim Q}}{Q \text{ face}}
\]

**Corollary:**
- Humbert-Hochin formula for \( S(G) \)
- New formula for \( S(H) \)
- Schmitt formula for \( S(P) \).