Matroids

Matroid is a combinatorial model of independence.

A matroid \( M = (E, \mathcal{I}) \) is a set \( E \) with a collection \( \mathcal{I} \) of subsets of \( E \), called "independent sets," such that

- \( \emptyset \in \mathcal{I} \)
- If \( I \subseteq J \) and \( J \in \mathcal{I} \) then \( I \in \mathcal{I} \)
- If \( I, J \in \mathcal{I} \) and \( |I| < |J| \) then there exists \( i \in J - I \) such that \( I \cup \{i\} \in \mathcal{I} \).

A basis is a maximal independent set.
The collection \( \mathcal{B} \) of bases determine \( M \).

**Ex 1.**
\[ E = \{a, b, c, d, e, f\} \]
\[ \mathcal{I} = \{\emptyset, a, b, c, d, e, ab, ac, ad, ae, bc, bd, be, cd, ca, \]
\[ ab, ac, bd, b, ce, acd, ace \} \]
\[ \mathcal{B} = \{abc, abd, ace, acd, ace \} \]

**Ex 2** (Linear matroids)
\( E \) = set of vectors in a vector space
\( \mathcal{I} \) = linearly independent subsets.

**Ex 3** (Graphical matroids)
\( E \) = edges of a graph
\( \mathcal{I} \) = subsets of \( E \) with no cycles.

**Ex 4** (Algebraic matroids)
\( E \) = elements of a field extension over \( \mathbb{F} \)
\( \mathcal{I} \) = subsets of \( E \) algebraically independent over \( \mathbb{F} \).

- If \( \mathbb{F} = \mathbb{F}(x, y, z) \)
- \( a = x + y + z \)
- \( b = x + y \)
- \( c = x - y \)
- \( d = xy \)
- \( e = x^2y^2 \)
- \( f = 1 \)

Many other examples!
If $M = (E, B)$, $M' = (E', B')$ are matroids, then the direct sum $M \oplus M'$ has
- ground set $E \cup E'$
- bases: $\{ B \cup B' : B \subseteq B, B' \subseteq B' \}$

Fact: $M \oplus M'$ is a matroid.

**Prop.** The product $M \cdot M' = M \otimes M'$ and coproduct
\[ \Delta(M) = \sum_{A \subseteq E} (M/A) \otimes (M/A) \]
give a Hopf algebra of matroids.

For graphical matroids, this is essentially the same as the (second) Hopf algebra of graphs.

If $A = \{ a_1, \ldots, a_k \}$ then
- $M/A = M/a_1/a_2/\ldots/a_k$
- $M \setminus A = M \setminus (E - A)$