Ex 5. (Incidence Hopf alg. of lattices)

A lattice is a poset $L$ such that:

- any two elements $a, b \in L$ have a join $a \lor b$ and a meet $a \land b$

such that:

- $a \lor b \geq a$
- $a \lor b \geq b$
- $a \land b \leq a$
- $a \land b \leq b$

If $c \leq a, c \leq b$ then $c \leq a \lor b$
If $d \leq a, d \leq b$ then $d \leq a \land b$

Least upper bound $\uparrow$

Greatest lower bound $\downarrow$

Examples:

No: $0 \lor c$

Yes: $\bigcirc$

$b \lor c$?

Other examples:

- $(2^5, \leq) \quad A \lor B = A \lor B, \quad A \land B = A \land B$
- $(\mathbb{N}, \text{divisibility}) \quad a \lor b = \gcd(a, b), \quad a \land b = \text{lcm}(a, b)$
- (Subgroup of $\mathbb{G}, \leq) \quad H \lor H' = H \land H', \quad H \lor H' = \langle H, H' \rangle$

Ex 6. (Another Hopf algebra of posets)

A lattice $L$ is distributive if $\land$ and $\lor$ are:

\[
\begin{align*}
\land & (a \land (b \lor c)) = (a \land b) \lor (a \land c) \\
\lor & (a \lor (b \land c)) = (a \lor b) \land (a \lor c)
\end{align*}
\]

Easy facts/exercises

- Either property implies the other one
- Distributive lattices are closed under taking subintervals, products.

$\Rightarrow$ There is an incidence Hopf alg. of distributive.
Why is the incidence Hopf alg of distributive lattice nice to have? Because distributive lattices have a lot of structure:

Let $P$ be a poset and $J(P) = \{ \text{downsets of } P \}$ such that $Q \subseteq P$ such that $x \leq y, x \in Q \Rightarrow y \in Q$.

Then $(J(P), \subseteq)$ is distributive $A \vee B = A \cup B, \quad A \wedge B = A \cap B$

Ex: $P = \{1, 2, 3, 4, 5\}$

$J(P) = \{1, 2, 3, 4, 5\}$

**Fundamental Theorem for Finite Distributive Lattices**

Let $L$ be a finite distributive lattice. There is a unique (up to isomorphism) poset $P$ such that $L \cong J(P)$

(Birkhoff, 1947)

**Sketch of Proof:**

Existence:

Say $p \in L$ is join-irreducible if there are no $a, b \in L$ such that $a \wedge b = p$.

Let $P$ be the join-irreducible elements of $L$.

Let $P$ inherit the partial order from $L$.

Claim: $L \cong J(P)$

Map: $\phi: L \rightarrow J(P)$

$t \mapsto f(t) = \{ s \in P : s \leq t \}$

$L \leftarrow J(P) : \phi^{-1}$

$V_i \leftarrow I$

(Check details.)

Uniqueness:

Claim: The poset of join-irreducibles of $J(P)$ is isomorphic to $P$.

$\phi$: There is a bijection $\{ \text{downsets of } J(P) \} \rightarrow D_{\text{max}} = \{ \text{maximal chains} \}$

$\phi^{-1}$: There is a bijection $D_{\text{max}} = \{ \text{maximal chains} \} \rightarrow \{ \text{maximal downsets} \}$

$P_{\text{max}} = \{ p : p \text{ is a join-irreducible} \}$

So the join-irreducibles of $J(P)$ are those $D_{\text{max}}$.

Then $J(P) \cong J(Q) \Rightarrow P \cong Q$.
So basis elt of $H(G)$ are

$\equiv$ classes of distributive lattices $L$

$\equiv$ classes of posets $P$

Product: $L_1 \circ L_2 = L_1 \times L_2$

$\Rightarrow$

$P_1 \circ P_2 = P_1 \cup P_2$

$L_1 = J(R), L_2 = J(R_2)$

Coproduct: $\triangle (L) = \sum_{x \in L} [\hat{0}, x] \otimes [x, \hat{1}]$

$\Rightarrow$

$\triangle (P) = \sum_{D \text{ daunted}} D \otimes (P \downarrow D)$

$L \equiv J(P)$

$[\hat{0}, x] \equiv J(D)$

$[x, \hat{1}] \equiv J(P \downarrow D)$

Antipode: $S(L) = \sum (-1)^n [x_0, x_1] \times \cdots \times [x_{n-1}, x_n]$

$\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$

$\Rightarrow$

$S(P) = \sum (-1)^n [f^{-1}(1) \cup \cdots \cup f^{-1}(n)]$

$f: P \rightarrow [n]$

surjective, order preserving

$\left(\begin{array}{c}
\{x_3 \text{ in } J(P)\} \\
\{D_3 \text{ daunted in } P\} \\
\therefore P \rightarrow [3] \\
f(P \backslash D_3) = \hat{0}
\end{array}\right)$