Ideals, Quotients

If \( B \) is a bialgebra, a biideal \( \mathcal{I} \subseteq B \) is a subset which is a two-sided ideal and a two-sided coideal.

In that case, \( B/\mathcal{I} \) is a quotient bialgebra, which inherits the bialgebra structure from \( B \).

The 1st 1st from Thm holds.

If \( H \) is a Hopf algebra, a Hopf ideal is a biideal \( \mathcal{I} \) such that \( S(\mathcal{I}) \subseteq \mathcal{I} \).

In that case, \( H/\mathcal{I} \) is a quotient Hopf algebra which inherits its structure from \( H \).

The 1st 1st from Thm holds.

Dual

If \( (H, m, \epsilon, \Delta, S) \) is a finite-dimensional Hopf algebra, then \( (H^*, \Delta^*, \epsilon^*, m^*, \epsilon^*, \Delta^*) \) is the dual Hopf algebra.

Often the antipode comes for free.

**Def** A bialgebra \( H \) is graded if

- \( H = \bigoplus_{n \geq 0} H_n \)
- \( H_i \cdot H_j \subseteq H_{i+j} \quad \forall i, j \geq 0 \)
- \( \Delta H_n \subseteq \bigoplus_{i+j=n} H_i \otimes H_j \quad \forall n \geq 0 \)
- \( \epsilon H_n = 0 \quad \forall n \geq 1 \)

It is connected if \( H_0 \cong K \)

**Ex.**

- \( K[x] = \bigoplus_{n \geq 0} \{x^n\} \), \( x^n x^j = x^{i+j} \), \( \Delta x^n = \sum_{i+j=n} x^i \otimes x^j \)
- \( K[x] \text{ isom. class of finite graphs} \implies H \)

\( H_n = \text{graph on } n \text{ vertices} \)

\( G_1 \cdot G_2 = \text{disjoint union} \)

\( \Delta (G) = \sum_{S \subseteq V} G|_S \otimes G|_{V-S} \) (Check that is a bialgebra)

- \( K \) perm. of some \( \{G_n\} = H \)

\( H_n = K S^n \)

\( \Pi_1 \cdot \Pi_2 = \sum \text{shuffles of } \Pi_1, \Pi_2 \)

\( \Delta (\Pi) = \sum \text{st}(\Pi_1 \ldots \Pi_S) \otimes \text{st}(\Pi_{n-S} \ldots \Pi_n) \)
Theorem (Takeuchi '71)

A graded connected bialgebra \( H \) has an antipode. If \( \pi = I - u e : H \to H \) then

\[
S = \sum_{n \geq 0} (-1)^n m^{n-1} \pi \otimes n \Delta^{n-1}
\]

which turns it into a Hopf algebra.

Convention: \( m^0 = \Delta^0 = id \), \( \Delta^n(h_m) = 0 \) for \( n > m \)

\( m^{-1} = u, \Delta^{-1} = e \).

So \( \Delta^n(h) \) is a finite sum for any \( h \in H \).

pf Recall that in the convolution product

\[
\pi^{*n} = \sum_{(h)} \pi(h_{(1)}) \cdots \pi(h_{(n)})
\]

\[
= m^{n-1} \pi \otimes n \Delta^{n-1}
\]

so really

\[
S = \sum_{n \geq 0} (-1)^n \pi^{*n}
\]

Then

\[
S \times I = \left[ \sum_{n \geq 0} (-1)^n \pi^{*n} \right] \times (I + u e)
\]

\[
= \sum_{n \geq 0} (-1)^n \pi^{*n} + \sum_{n \geq 0} (-1)^n \pi^{*n} = I \times S = u e
\]

Similarly \( I \times S = u e \).

Remark

- A graded bialgebra \( H \) is connected \( \iff u e |_{H_0} = I |_{H_0} \)

This is because we always have \( \text{ker}(H \to H_0) \to H_0 \)

\[
\xrightarrow{\text{ker}(H \to H_0)} \xrightarrow{\text{ker}(H \to H_0)}
\]

- if \( H \neq k \) then \( u = e^{-1} \)

- if \( \dim(H_0) > 1 \), \( \dim(\text{im}(u e)) = 1 < \dim(\text{im}(I)) \)

- On \( H_n (n \geq 1) \), \( u e = 0 \)

So \( (I - u e)(h) \) just drops the \( H_0 \) part of \( h \).

This formula isn't always so practical.