(b) Prove that the two-sided ideal \( \langle xy - 1, yx - 1 \rangle \) is a biideal of \( F \), and therefore the quotient \( H = F / \langle xy - 1, yx - 1 \rangle \) is a bialgebra.

**Proof.** \( J = \langle xy - 1, yx - 1 \rangle \) is an ideal by construction. Then we must verify that \( J \) is a coideal. First, we check that \( J \subseteq \ker \epsilon \). Given \( h \in J \), we have \( h = a(xy - 1) + b(yx - 1) + (xy - 1)c + (yx - 1)d \) for \( a, b, c, d \in F \). Then

\[
\epsilon(h) = \epsilon(a(xy - 1) + b(yx - 1) + (xy - 1)c + (yx - 1)d) = \epsilon(a)(\epsilon(xy) - \epsilon(1)) + \epsilon(b)(\epsilon(yx) - \epsilon(1)) + (\epsilon(xy) - \epsilon(1))\epsilon(c) + (\epsilon(yx) - \epsilon(1))\epsilon(d) = \epsilon(a)(1 - 1) + \epsilon(b)(1 - 1) + (1 - 1)\epsilon(c) + (1 - 1)\epsilon(d) = 0.
\]

Next, we verify that \( \Delta(J) \subseteq J \otimes F + F \otimes J \). Observe that
\[\Delta(xy - 1) = \Delta(x)\Delta(y) - \Delta(1)\]
\[= (x \otimes x)(y \otimes y) - 1 \otimes 1\]
\[= xy \otimes xy - 1 \otimes 1\]
\[= \frac{1}{2} [xy \otimes xy + xy \otimes 1 - 1 \otimes xy - 1 \otimes 1\]
\[+ xy \otimes xy - xy \otimes 1 + 1 \otimes xy - 1 \otimes 1]\]
\[= \frac{1}{2} [(xy - 1) \otimes (xy + 1) + (xy + 1) \otimes (xy - 1)]\]
\[\in J \otimes F + F \otimes J\]

This also works for \((yx - 1)\),

\[\Delta(yx - 1) = \Delta(x)\Delta(y) - \Delta(1)\]
\[= \frac{1}{2} [(yx - 1) \otimes (yx + 1) + (yx + 1) \otimes (yx - 1)]\]
\[\in J \otimes F + F \otimes J\]

Since \(J\) is an ideal, (as is \(F\)), then \(h \otimes k(\Delta(xy - 1)) \in J \otimes F + F \otimes J\) for any \(f, k \in F\), as is \((\Delta(xy - 1))h \otimes k\), and similarly for \(\Delta(yx - 1)\). Then, apply \(\Delta\) to an element of \(J\).

\[\Delta(a(xy - 1)b + c(yx - 1)d)\]
\[= \Delta(a)(\Delta(xy - 1))\Delta(b) + \Delta(c)(\Delta(yx - 1))\Delta(d)\]
\[\in F \otimes J + J \otimes F.\]

This verifies that \(J\) is a coideal, so \(J\) is a biideal. Thus \(H = F/J\) is a bialgebra.

\(\square\)

(c) Prove that \(H\) is a Hopf algebra by finding its (unique) antipode \(S\). Find the order of \(S\).

Proof.

\[
\begin{array}{c}
  x \otimes x \xrightarrow{S \otimes I} S(x) \otimes x \\
  \Delta \downarrow \quad \quad \uparrow m \\
  x \xrightarrow{\epsilon} 1 \xrightarrow{u} S(x)
\end{array}
\]

Thus, we see that \(S(x) = y\).
To calculate $S(z)$, we use the relation $xy = 1$ in the coproduct of $z$.

Since $xy = 1$, from the bottom equality $S(z) + zS(x) = 0$, we get $S(z) = -zy$. Plugging into the top equality, we get $z - zyx = z - z = 0$. Then we have a well defined antipode given by

$S(x) = y$
$S(y) = x$
$S(z) = -zy.$

We see that $S^2(x) = S(y) = x$, $S^2(y) = S(x) = y$. What remains is to compute the order of $S$ with respect to $z$:

$S(z) = -zy$
$S^2(z) = xzy$
$S^3(z) = -xzy^2$
$S^4(z) = x^2zy^2$

$\vdots$

$S^n(z) = (-1)^n x^{\left\lfloor \frac{n}{2} \right\rfloor} y^{\left\lceil \frac{n}{2} \right\rceil}.$

The computations here have been omitted for the sake of brevity. The key to understanding this formula is to note that the antipode...
is an antihomomorphism. That is, for \(a, b \in H\), \(S(ab) = S(b)S(a)\). Given this, and the fact that \(S(z) = -zy\) and \(S\) turns \(xs\) into \(ys\) and vice versa, we see that for a word of the form \(x^m z y^n\), 
\[S(x^m z y^n) = (S(y))^n S(z)(S(x))^m = -x^n z y^{m+1}\]
Then with each application of \(S\), all \(x\)s on the left become \(y\)s on the right, and vice versa, and we add one extra \(y\) on the right side of \(z\).
Since we are in a noncommutative algebra, and our only relations are \(xy = yx = 1\), we see that \((-1)^n x^{\lfloor \frac{n}{2} \rfloor} z y^{\lceil \frac{n}{2} \rceil} = z\) if and only if \(n = 0\). Then the order of \(S\) in \(H\) is infinite. 

(d) Prove that the two-sided ideal \(\langle x^n - 1 \rangle\) is a Hopf ideal of \(H\), and therefore \(J = H/\langle x^n - 1 \rangle\) is a Hopf algebra.

Proof. We check that \(K = \langle x^n - 1 \rangle\) is a biideal: Let \(h \in K\). Then \(h = a(x^n - 1) + (x^n - 1)b\) for some \(a, b \in H\).

\[
\epsilon(h) = \epsilon(a(x^n - 1) + (x^n - 1)b) \\
= \epsilon(a)(\epsilon(x^n) - 1) + (\epsilon(x^n) - 1)\epsilon(b) \\
= \epsilon(a)(0) + (0)\epsilon(b) \\
= 0,
\]
so \(K \subseteq \ker \epsilon\).

\[
\Delta(h) = \Delta(a(x^n - 1)b) \\
= \Delta(a)(\Delta(x^n) - \Delta(1))\Delta(b) \\
= \Delta(a)(x^n \otimes x^n - 1 \otimes 1)\Delta(b) \\
= \Delta(a) \left( \frac{1}{2} [(x^n - 1) \otimes (x^n + 1) + (x^n + 1) \otimes (x^n - 1)] \right) \Delta(b) \\
\in H \otimes K + K \otimes H.
\]
To show that the biideal is also a Hopf ideal, we need to show that \(S(K) \subseteq K\).

\[
S(h) = S(a(x^n - 1) + (x^n - 1)b) \\
= S(a)(S(x^n) - S(1)) + (S(x^n) - S(1))S(b) \\
= S(a)(y^n - 1) + (y^n - 1)S(b) \\
= S(a)(x^n - 1)(-y^n) + (x^n - 1)(-y^n)S(b) \\
\in K.
\]
This completes the verification that \(K\) is a Hopf ideal, so \(H/K\) is a Hopf algebra. 

(e) Prove that the antipode of \(J\) has order \(2n\).
Proof. Note that our calculations from part (c) still hold after modding out by \( K \), so we still have

\[
S(x) = y \\
S(y) = x \\
S(z) = -zy.
\]

We still have \( S^2(x) = x \) and \( S^2(y) = y \), so we need to recalculate the order of \( S \) on \( z \). Our relations are \( xy = yx = x^n = 1 \). Also, we can use the identity \( yx = 1 \) and \( x^n = 1 \) to get \( 1 = y^n \).

\[
S^k(z) = (-1)^k x^\left\lfloor \frac{k}{2} \right\rfloor y^\left\lceil \frac{k}{2} \right\rceil \\
\overset{\text{set}}{=} z.
\]

Then

\[
(-1)^k = 1, \\
x^\left\lfloor \frac{k}{2} \right\rfloor = 1, \\
y^\left\lceil \frac{k}{2} \right\rceil = 1.
\]

By \((-1)^k = 1\), we know \( k = 2m \) for some \( m \in \mathbb{Z} \), so \( \left\lfloor \frac{k}{2} \right\rfloor = \left\lceil \frac{k}{2} \right\rceil = m \). Then we are looking for the least \( m \) such that \( x^m = y^m = 1 \). This gives \( m = n \), so the order of \( S \) on \( z \) is \( 2n \). Since \( S^2(x) = x \) and \( S^2(y) = y \), then \( S^{2n}(x) = x \) and \( S^{2n}(y) = y \), so the order of \( S \) is \( 2n \).

\( \square \)

(5) (Hopf algebras of permutations and graphs.)

(a) Recall that we defined the \textit{Hopf algebra of graphs} on the linear span of the isomorphism classes of finite simple graphs. The product \( G \cdot H \) is the disjoint union of \( G \) and \( H \). The coproduct of a graph \( G \) on vertex set \( V \) is \( \Delta(G) = \sum_{S \subseteq V} G|_S \otimes G|_{V \setminus S} \). Here \( G|_A \) denotes the induced subgraph of \( G \) with vertex set \( A \). The unit is given by \( u(1) = \emptyset \), the graph with no vertices. The counit is given by \( \epsilon(\emptyset) = 1 \) and \( \epsilon(G) = 0 \) for all \( G \neq \emptyset \). Vertify that this is indeed a Hopf algebra. Find a simple formula for the antipode.

Proof. To verify that this is a Hopf algebra, we must check the following properties: associative multiplication coassociative coalgebra morphisms, unitary and counitary properties, \( m \) and \( u \) coalgebra morphisms, and a well defined antipode.

Associative multiplication is trivial, since multiplication is given by the disjoint union. \((A \cdot B) \cdot C = A \cdot (B \cdot C)\) for all disjoint sets \( A, B \) and \( C \).