Math 850: Homework 3

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Problem 1. Let $P$ be a finite poset and $A(P)$ be its incidence algebra. Recall that the identity is the function $1 \in A(P)$ defined by $1([x,x]) = 1$ for all $x$ in $P$ and $1([x,y]) = 0$ for all $x < y$ in $P$. Define the zeta function of $P$ by $\zeta([x,y]) = 1$ for all $x \leq y$ in $P$.

(a) Prove that $f \in A(P)$ has a two-sided inverse if and only if $f([x,y]) \neq 0$ for all $x \in P$.

(b) By part (a), $(2 - \zeta)$ is invertible in $A(P)$. Prove that

$$(2 - \zeta)^{-1}([x,y]) = (\# \text{ of chains in } P \text{ from } x \text{ to } y).$$

(c) By part (a), $\zeta$ is invertible in $A(P)$. Its inverse is $\mu = \zeta^{-1}$ is call the Mobius function of $P$. Prove the Mobius Inversion formula:

Let $f : P \to V$ and $g : P \to V$ be functions for $P$ to a vector space $V$. Then

$$g(y) = \sum_{x \leq y} f(x) \quad \text{for all } y \in P$$

if and only if

$$f(y) = \sum_{x \leq y} \mu(x,y) g(x) \quad \text{for all } y \in P$$

Solution. Crista Moreno, Karla Lanzas, Nina Cerutti, Stephen Collazos

(a) Suppose $f$ has a two-sided inverse $g$. Then $f \ast g = 1 = g \ast f$ so that

$$f \ast g([x,x]) = f([x,x]) g([x,x]) = 1([x,x]) = 1.$$ 

Therefore $f([x,x]) \neq 0$.

Conversely suppose $f([x,x]) \neq 0$. I will show that we can define a function $g$ such that $f \ast g([x,y]) = 1([x,y])$ for any $x \leq y \in P$ by induction on the length of the maximal chain in $[x,y]$. For any $x \in P$, let

$$g([x,x]) = \frac{1}{f([x,x])}.$$ 

Then

$$f \ast g([x,x]) = f([x,x]) g([x,x]) = f([x,x]) \frac{1}{f([x,x])} = 1 = 1([x,x]).$$
Similarly $g \circ f([x,y]) = 1$. Now for an interval $[x,y]$ where $y$ covers $x$, let $g([x,y]) = \frac{-f([x,y])}{f([x,y])f([y,y])}$.

We can define $g$ in this way because $f([x,x])f([y,y]) \neq 0$. Then

$$f \circ g([x,y]) = f([x,x])g([x,y]) + f([x,y])g([y,y])$$
$$= f([x,x])\frac{-f([x,y])}{f([x,x])f([y,y])} + f([x,y])\frac{1}{f([y,y])} = 0$$
$$= 1([x,y])$$

and similarly $g \circ f([x,y]) = 0$. Suppose we can define $g$ for any interval with maximal chain of length $n$. Consider an interval $[x,y]$ with a maximal chain of length $n + 1$. Then let $g([x,y]) = \frac{\sum_{x \leq z \leq y} f([x,z])g([z,y])}{f([x,x])}$.

Then

$$f \circ g([x,y]) = \sum_{x \leq z \leq y} f([x,z])g([z,y])$$
$$= f([x,x])g([x,y]) + \sum_{x \leq z \leq y} f([x,z])g([z,y]) = 0$$
$$= 1([x,y])$$

and similarly $g \circ f([x,y]) = 0$. Therefore, $g$ is a two-sided inverse of $f$.

(b) First consider the function $(\zeta - 1)$:

$$(\zeta - 1)([x,y]) = \begin{cases} 1 & x < y \\ 0 & x = y \end{cases}$$

For positive integers $k$ we see that,

$$(\zeta - 1)^k([x,y]) = \sum_{x = x_0 < x_1 < \cdots < x_k = y} \prod_{i=0}^{k-1} (\zeta - 1)([x_i, x_{i+1}])$$
$$= \sum_{x = x_0 < x_1 < \cdots < x_k = y} 1$$

is the number of chains of length $k$ in $[x,y]$.

Now let $l$ be the length of the maximal chain of $[x,y]$. Then certainly, $(\zeta - 1)^{l+1}([u,v]) = 0$ for all $x \leq u \leq v \leq y$ because $[x,y]$ has 0 chains of length $l + 1$. Therefore we can observe

$$(2 - \zeta)\left(1 + (\zeta - 1) + (\zeta - 1)^2 + \cdots + (\zeta - 1)^l\right)([x,y])$$
$$= (1 - (\zeta - 1))\left(1 + (\zeta - 1) + (\zeta - 1)^2 + \cdots + (\zeta - 1)^l\right)([x,y])$$
$$= (1 - (\zeta - 1)^{l+1})([x,y]) = 1([x,y])$$

This shows that $(2 - \zeta)^{-1} = 1 + (\zeta - 1) + (\zeta - 1)^2 + \cdots (\zeta - 1)^l$. In other words, $(2 - \zeta)^{-1}$ is the number of chains of length 0, the number of chains of length 1, the number of chains of length 2, \cdots, the number of chains of length $l = \text{the total number of chains in } P \text{ from } x \text{ to } y$. 

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(c) Suppose \( g(y) = \sum_{x \leq y} f(x) \) for all \( y \in P \). Then

\[
\sum_{x \leq y} \mu([x,y])g(x) = \sum_{x \leq y} \mu([x,y]) \left( \sum_{z \leq x} f(z) \right) = \sum_{z \leq y} \mu([z,y])f(z)
\]

for all \( z \leq x \in P \) because \( \zeta([z,x]) = 1 \) for all \( z \leq x \in P \)

\[
= \sum_{z \leq y} f(z) \left( \sum_{z \leq x \leq y} \zeta([z,x])\mu([x,y]) \right)
\]

\[
= \sum_{z \leq y} f(z) \left( \sum_{z \leq x \leq y} \frac{\mathbf{1}([z,x])}{\mu([x,y])} \right)
\]

\[
= \begin{cases} 
1 & z = y \\
0 & z < y 
\end{cases}
\]

\[
= f(y)
\]

Conversely, suppose \( f(y) = \sum_{x \leq y} \mu(x,y)g(x) \). Then

\[
\sum_{x \leq y} f(x) = \sum_{x \leq y} \mu([z,x])g(z)
\]

\[
= \sum_{z \leq y} \mu([z,x])\zeta([x,y])g(z)
\]

\[
= \sum_{z \leq y} \mathbf{1}([z,y])g(z)
\]

\[
= \begin{cases} 
1 & z = y \\
0 & z < y 
\end{cases}
\]

\[
= g(y)
\]

**Problem 2.** Let \( \mathbb{H} \) be the quaternions, regarded as a 4-dimensional algebra over \( \mathbb{R} \). Let \( C = \mathbb{H}^* \) be the dual coalgebra. Find the cocommutative element \( a \in C \) such that the subcoalgebra of \( C \) generated by \( a \) is not cocommutative.

**Solution.** Crista Moreno, Karla Lanzas, Nina Cerutti, Karen Walters, Sweedler

For each \( x,y \in \mathbb{H} \), let \( L(xy) = -xy \) so that \( L \) is a \( \mathbb{R} \)-linear map from \( \mathbb{H} \) to \( \mathbb{H} \). We can define \( a \in A \) by \( \langle a, xy \rangle = L \). We see that \( \langle a, 1 \rangle \neq 0 \), otherwise \( L(xy) \) would be 0, so \( a \neq 0 \). Now assume that

\[
\Delta(a) \sum_i x_i^* \otimes y_i^*.
\]

For all \( x,y \in \mathbb{H} \) we have,

\[
\sum_i x_i^* \otimes y_i^* = \langle \Delta(a), x \otimes y \rangle
\]

\[
= \langle a, xy \rangle
\]

\[
= \langle a, yx \rangle
\]

\[
= \langle \Delta(a), y \otimes x \rangle
\]

\[
= \sum_i \langle x_i^*, y \rangle \langle y_i^*, x \rangle.
\]