4. A Non-Commutative, Non-Cocommutative Bialgebra: Let \( q \in \mathbb{F} \) be non-zero. Consider the \( \mathbb{F} \)-algebra \( H_4 \) generated by indeterminates \( g \) and \( x \) subject to the relations \( g^2 = 1, x^2 = 0 \), and \( xg = -gx \).

(a) Show that \( 1, g, x, gx \) form a basis for \( H_4 \).

**Proof.** We begin by showing that \( H_4 \in \text{span}\{1, g, x, gx\} \). Note that \( \text{span}\{gx\} = \text{span}\{xg\} \) because in an algebra we can freely scale by elements of the field, for example -1. Thus we can switch between \( xg \) and \( gx \) without altering the span of the set. This allows us to transform any expression composed of \( x \) and \( g \) into the form \( g^i x^j \). Note that with the relations \( x^2 = 0 \) and \( g^2 = 1 \), we have \( x^j = 0 \) for any \( j > 1 \), and \( g^i = g \) only for odd \( i \), 1 otherwise. Thus the only possible unique (relative to the field) terms we can build from \( x \) and \( g \) are \( 1, g, x, \) and \( gx \). Thus \( H_4 \in \text{span}\{1, g, x, gx\} \).

To show linear dependence in these terms, assume we have some relation \( \lambda_1 1 + \lambda_2 g + \lambda_3 x + \lambda_4 gx = 0 \) where not all \( \lambda \) equal zero. If this were the case, such a relation would have to be given in the construction of the algebra. Since no such relation is given, we must have all \( \lambda = 0 \), which establishes linear independence. Thus, \( 1, g, x, gx \) form a basis for \( H_4 \). \( \square \)

(b) Express the product \( (a + bg + cx + dgx)(a' + b'g + c'x + d'gx) \) in terms of this basis, where \( a, b, c, d, a', b', c', d' \in \mathbb{F} \).

**Proof.**

\[
(a + bg + cx + dgx)(a' + b'g + c'x + d'gx) = a a' + a b' g + a c' x + a d' g x + b a' g + b b' g^2 + b c' g x + b d' g^2 x + ... \\
+ c a' x + c b' x g + c c' x^2 + c d' x g x + d a' g x + d b' g x g + d c' g x^2 + d d' g x g x \\
= a a' + a b' g + a c' x + a d' g x + b a' g + b b' g + b c' g x + b d' x + ... \\
+ c a' x - c b' g x + 0 + 0 + d a' g x - d b' x + 0 + 0 \\
= (a a' + b b') + (a b' + b a') g + (a c' + b d' + c a' - d b') x + (a d' + b c' - c b' + d a') g x
\]

\( \square \)