homework one . due tuesday feb 7 at 11:59pm (sf time = bog time – 3:00)

Note. You are encouraged to work together on the homework, but you must state who you worked with in each problem. You must write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)
You must turn in your homework – in one .pdf file by email at hopfcombinatorics@gmail.com.

In the problems below, \( F \) denotes an arbitrary field and \( G \) denotes an arbitrary group.

1. (Bases of tensor products.)
   Let \( V \) and \( W \) be \( F \)-vector spaces. If \( \{ v_i : i \in I \} \) is a basis for \( V \) and \( \{ w_j : j \in J \} \) is a basis for \( W \), prove that \( \{ v_i \otimes w_j : i \in I, j \in J \} \) is a basis for \( V \otimes W \). Conclude that, if \( \dim V \) and \( \dim W \) are finite, then \( \dim V \otimes W = \dim V \dim W \).

2. (Some examples of \( F \)-algebras.)
   (a) Prove that \( F[x] \otimes F[y] \cong F[x, y] \) as \( F \)-algebras.
   (b) If \( A \) is a \( F \)-algebra, let \( M_{n \times n}(A) \) be the \( F \)-algebra of \( n \times n \) matrices with entries in \( A \). Explain briefly why \( M_{n \times n}(A) \) is a \( F \)-algebra. Prove that \( M_{n \times n}(A) \cong M_{n \times n}(F) \otimes A \) as \( F \)-algebras.
   (c) Prove that \( M_{n \times n}(F) \otimes M_{n \times n}(F) \cong M_{m \times m}(F) \) as \( F \)-algebras.

3. (Group-like elements in the group ring.)
   In a coalgebra, we say the element \( x \) is group-like if \( \Delta(x) = x \otimes x \). Prove that in the group ring \( F[G] \), \( x \) is grouplike if and only if \( x \in G \).

4. (A non-commutative, non-cocommutative bialgebra.)
   Let \( q \in F \) be nonzero. Consider the \( F \)-algebra \( H_4 \) generated by indeterminates \( g \) and \( x \) subject to the relations \( g^2 = 1, x^2 = 0, \) and \( xg = -gx \).
   (a) Show that \( 1, g, x, \) and \( gx \) form a basis for \( H_4 \).
   (b) Express the product \( (a + bg + cx + dgx)(a' + b'g + c'x + d'gx) \) in terms of this basis, where \( a, b, c, d, a', b', c', d' \in F \).
   (c) Show that the coproduct given by \( \Delta g = g \otimes g \) and \( \Delta x = x \otimes 1 + g \otimes x \) and the counit given by \( \epsilon(g) = 1 \) and \( \epsilon(x) = 0 \) turn \( H_4 \) into a bialgebra.
   (d) Express the coproduct \( \Delta(a + bg + cx + dgx) \) in terms of this basis.

5. (Tensor, symmetric, and exterior algebras.)
   Let \( V \) be a given \( d \)-dimensional \( F \)-vector space for some \( d \in N \).
   (a) Let \( V^\otimes k = V \otimes \cdots \otimes V \) (where there are \( k \) copies of \( V \)) be the \( k \)th tensor product of \( V \), and let
   \[
   T(V) := \bigoplus_{k=0}^{\infty} V^\otimes k = F \oplus V \oplus V^\otimes 2 \oplus V^\otimes 3 \oplus \cdots
   \]
   be the tensor algebra of \( V \). Find its Hilbert series \( \text{Hilb}(T(V); q) = \sum_{n=0}^{\infty} \dim(V^\otimes k)q^k \).
   (b) Let
   \[
   S(V) = T(V)/\langle u \otimes v - v \otimes u : u, v \in V \rangle
   \]
   be the symmetric algebra of \( V \). Find its Hilbert series.
   (c) Let
   \[
   S(V) = T(V)/\langle v \otimes v : v \in V \rangle
   \]
   be the exterior algebra of \( V \). Find its Hilbert series.
6. (Bonus problem 1: Finite generation of algebras.)

Find an example of two \( F \)-algebras \( A \subset B \) such that \( B \) is finitely generated and \( A \) is not finitely generated.

(We say an algebra \( A \) over \( F \) is finitely generated if there exist \( a_1, \ldots, a_n \in A \) such that \( A \) is generated as a ring by \( a_1, \ldots, a_n \), and \( F \).)

7. (Bonus problem 2: A “Catalan algebra”.)

Let \( q \in F \) be nonzero. Consider the \( F \)-algebra \( A_n \) with generators \( e_1, \ldots, e_{n-1} \) subject to the relations

\[
\begin{align*}
e_i^2 &= e_i & \text{for all } i, \\
e_i e_j &= e_j e_i & \text{for all } i, j \text{ with } |i - j| > 1, \\
e_i e_j e_i &= q e_i & \text{for all } i, j \text{ with } |i - j| = 1.
\end{align*}
\]

Prove that the dimension of \( A_n \) is the \( n \)th Catalan number \( C_n = \binom{2n}{n} / (n+1) \).