Proof of Dehn-Sommerville relations:

Recall the zeta polynomial

\[ Z(L, n) = \mathcal{L}^n (\hat{\theta}, \hat{\varphi}) \]

\[ = \sum_{\rho \leq \beta \leq \mathcal{L} \rho \beta} \mathcal{L} (\hat{\theta}, \hat{\varphi}) \mathcal{L} (\rho, \beta) \mathcal{L} (\beta, \rho) \mathcal{L} (\beta, \beta) \]

\[ Z(L, n) = \# \text{ of multichains of length } n \text{ from } \hat{\theta} \] 

**Now**

\[ Z(L, n) = \mathcal{L}^{n-1} (\hat{\theta}, \hat{\varphi}) \]

\[ = \mathcal{L} (\hat{\theta}, \hat{\varphi}) \]

\[ = \sum_{\delta \leq \beta \leq \mathcal{L} \delta \beta} \mathcal{L} (\hat{\theta}, \hat{\varphi}) \mathcal{L} (\delta, \beta) \mathcal{L} (\beta, \delta) \mathcal{L} (\beta, \beta) \]

\[ = \sum_{\delta \leq \beta \leq \mathcal{L} \delta \beta} (-1)^{\delta} \mathcal{L} (\hat{\theta}, \hat{\varphi}) \mathcal{L} (\delta, \beta) \mathcal{L} (\beta, \delta) \mathcal{L} (\beta, \beta) \]

\[ = (-1)^{d} (\# \text{ of multichains of length } n \text{ from } \hat{\theta} \] 

\[ \# = (-1)^{d} Z(L, n) \]

Also

\[ Z(L, n) = \sum_{m=0}^{n} Z(L, \hat{\theta}, m) \text{ (we use } n-m \text{ time)} \]

So

\[ Z(L, n) - Z(L, n-1) = Z(L, \hat{\theta}, n) \]

Also

\[ Z(L, \hat{\theta}, n) = \sum_{i} f_{i} Z(F_{i}, n-1) \]

\[ = \sum_{i} f_{i} (n-1)^{i} \]

\[ \Rightarrow \text{ let the last face } F \in \text{ variable } \{0, \ldots, \dim \text{ then } } E_{0, F} = B_{i} \]

So

\[ \sum_{i} f_{i} (n-1)^{i} = (-1)^{d} Z(L, \hat{\theta}, n) \]

\[ \Rightarrow \sum_{i} (-1)^{i} f_{i} (n-1)^{i} = (-1)^{d} Z(L, \hat{\theta}, n-1) \]

Hence

\[ (-1)^{d} \sum_{i} f_{i} (n-1)^{i} = (-1)^{d} (-1)^{d} \sum_{i} f_{i} (-1)^{d-i} \]

\[ = \sum_{i} (-1)^{i} f_{i} (n-1)^{i} \]

\[ \sum_{i} h_{k} \left( \frac{1}{n} \right) d x = \sum_{i} h_{k} \left( \frac{1}{n+1} \right) d x \]

\[ h(x) = x^{d} h(1/x) \]

\[ h_{i} = h_{d-i} \]

To prove there are no other, need to construct enough polytopes to "span" all other fractals.

This characterizes the equality satisfied by the fractals of a simplicial polytope

Remarkably, the celebrated g-theorem characterizes exactly which vectors are fractals of simplicial polytopes. (McMullen 70, Billera-Lee 79, Stanley 79)
What if we count not only the four but also their incidences?

The flag f-vector of $P$ is

$$f(S) = \# \text{ of flags of form } F_1 \subset \cdots \subset F_k$$

of dimension $\{a_1, \ldots, a_k \mid a_k = 3\}$.

**Ex:**

- $f(\emptyset) = 1$
- $f(\{a\}) = 6$
- $f(\{a, b\}) = 12$
- $f(\{a, b, c\}) = 8$
- $f(\{a, b, c, d\}) = 24$
- $f(\{a, b, c, d, e\}) = 48$

Encode this in the non-commutative polynomial:

$$X_P(a, b) = aaa + 6aab + 12aba + 8aab$$
$$+ 24bba + 24bab + 24abb + 48bbb$$

Let the $ab$-index of $P$ be

$$\Psi_P(a, b) = X_P(a-b, b)$$

$$= 1 aaa + 5 aba + 11aba + 7 aab$$
$$+ 7 bba + 11 bab + 5 abb + 1 bbb$$

and let the $ab$-coefficients be the flag f-vector of $P$.

**Easy:**

$$h(S) = \sum_{T \in S} (-1)^{|T|-1} f(T)$$

$$f(S) = \sum_{T \in S} h(T)$$

Are there further relations among these?

For example, $h(S) = h(\emptyset)$?

Note: $\Psi_P(a, b) = (ab)^3 + 6(ab)(abab) + 4(abab)(ab)$

**Theorem.** (H. Bayer, L. Billera) "cd-index" - $cd$-index

For every polytope (or Eulerian poset) there exists a polynomial $\Phi_P(c, d)$ in non-commutative variables $c, d$ such that

$$\Psi_P(a, b) = \Phi_P(a+b, abab)$$

(Also, this determines all the relations among the $f(S)$)

The $0=2^3$ entries of the flag f-vector of a 3-polytope depends only on the 3 coefs of

$$c^3 \quad cd \quad dc$$

Because $\deg c = 1, \deg d = 2$, then all $F_{mn}$ cd-binomial of degree $n$. So:

The $2^n$ entries of the flag f-vector of a polytope are determined completely by $F_{nn} \approx 1.61^n$ of the entries, and no fewer!