Prop. \( P \) is a set, \( m \in \mathbb{N} \). Then we equal:

1. \( \# \) of order-preserving maps \( P \to m \)
2. \( \# \) of multichain \( \hat{D} = t_0 \leq t_1 \leq \ldots \leq t_m \) in \( J(P) \)
3. \( \# \) of length \( m \)
4. \( |J(P \times (m-1))| \)

\[ \text{Pf: } \begin{align*}
\hat{D} & = \hat{C}^3 \times \hat{C}^3 \times \hat{C}^3 = \hat{C}^6
\end{align*} \]

The \( \zeta \) polynomial of \( P \) is given by

\[ Z(P,m) = \# \text{ of multichain } t_1 \leq t_2 \leq \ldots \leq t_m \text{ in } P \]

This is indeed a polynomial:

\[ Z(P,m) = \sum_{i \geq 2} b_i \binom{n-2}{i-2} \]

where \( b_i \) is the \( i \)-th coefficient of the polynomial in \( n \) of degree \( i-2 \).

\[ \text{deg } Z(P,m) = \text{ht}(P) \]

The order polynomial of \( P \) is given by

\[ \Omega_P(m) = \# \text{ of order-preserving maps } P \to m = Z(J(P),m) \]

\[ \text{deg } \Omega_P(m) = |P| \]

leading coeff. = \( e(P)/|P|! \)

Thus there is a new algebraic approach.

Prop. \( \Omega \), if \( \rho_1 \rho_2 \ldots \rho_k \) are the max elements of \( P \),

\[ e(P) = e(P,\rho_1) \times \ldots \times e(P,\rho_k) \]

\( \# \) of lin. ext. \( v \) where \( v(\rho_i) = n = e(P,\rho_i) \)

Core \( e(P) \) can be computed by applying the "generalized Pascal recurrence" on \( J(P) \)

Ex. \( e(M) = 2^2 \times 2^2 \times 2^2 = 16 \)
Incidence Algebra

The incidence algebra $I(P)$ of a poset $P$ is the $R$-algebra of functions

$$f : I(P) \rightarrow R$$

(Where $I(P) = \{ [x, y] : x \leq y \text{ in } P \}$ with multiplication

$$fg(1, 2) = \sum_{x \leq y} f(x, y) g(y, 2)$$

This ring has a multiplicative identity

$$1(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

The zeta function of $P$ is

$$\zeta(x, y) = 1 \text{ for all } x, y \in P$$

Then

$$\zeta^2(x, y) = \sum_{x \leq z \leq y} \zeta(x, z) \zeta(z, y) = \sum \tau : [x, y]$$

and

$$\zeta^n(x, y) = \sum_{x \leq z_1 \leq \cdots \leq z_n \leq y} \zeta(x, z_1) \zeta(z_1, z_2) \cdots \zeta(z_{n-1}, y)$$

$$= \sum \tau : [x, y]$$

$$= \sum \tau : [x, y]$$

Similar, since

$$(\zeta^n(x, y)) = \begin{cases} 1 & x \leq y \\ 0 & x = y \end{cases}$$

we have

$$(\zeta^n(x, y)) = \# \text{ of } \tau : [x, y].$$

Prop: The following are equivalent for $f \in I(P)$:

- $f$ has a left-inverse
- $f$ has a right-inverse
- $f$ has a two-sided inverse
- $f(s, s) \neq 0 \text{ for all } s \in P.$

Proof: See book.

Since $(\zeta, x, y) = \begin{cases} 1 & x \leq y \\ 0 & x = y \end{cases}$

Then

$$(\zeta, x, y) = \# \text{ of } \tau : [x, y].$$

Proof: (2.7)

$$(\zeta, x, y) = (1 - (x, y - 1))^n = 1 + (x, y - 1)(x, y - 2)^2 + \cdots + (x, y - 1)^n$$

Note: $\zeta$ is invertible so led the Möbius function of $P$ be

$$M = \zeta^{-1}$$

Equivalently,

$$M(x, y) = \begin{cases} \sum_{x \leq z \leq y} \zeta(x, z) & x = y \\ 0 & x \neq y \end{cases}$$

Ex: $D_5$:

$$M(3, 5):$$

$$0 \quad 0 \quad -1 \quad 0 \quad 0$$

$$-1 \quad -1 \quad 1 \quad 0 \quad -1$$

$$0 \quad 0 \quad 0 \quad 1 \quad 0$$

$$0 \quad 1 \quad 0 \quad 0 \quad 0$$

$$1 \quad 0 \quad 0 \quad 0 \quad 0$$
Möbius Inversion Formula

Let \( P \) be a poset.

Let \( f: P \to \mathbb{R} \) be such that

\[
g(t) = \sum_{s \leq t} f(s) \quad \text{for all } t \in P
\]

Then

\[
f(t) = \sum_{s \leq t} \mu(s,t) g(s)
\]

1st Pf: For any \( t \),

\[
\sum_{s \leq t} \mu(s,t) \left( \sum_{r \leq s} g(r) \right) = \sum_{r \leq t} g(r) \sum_{r \leq s} \mu(s,t)
\]

\[
= \sum_{r \leq t} g(r) \left[ \sum_{r \leq s} \mu(s,t) \right]
\]

\[
= \sum_{r \leq t} g(r) \cdot \mu(r,t) = g(t) \quad \square
\]

2nd Pf: \( g = \leq f \iff f = \mu\cdot g \)

For details, see book. \( \square \)

Our the next few classes we will discuss Möbius functions and inversion more slowly and combinatorially. They are very important.