Posets

A (partially ordered set) is a set $P$ with a binary relation $\leq$ such that:

- reflexive: $x \leq x$ for $x \in P$
- antisymmetric: $x \leq y$ and $y \leq x$ imply $x = y$ for $x, y \in P$
- transitive: $x \leq y$ and $y \leq z$ imply $x \leq z$ for $x, y, z \in P$

Say $y$ covers $x$ ($x \leq y$) if $x < y$ and $x \neq y$.

Draw $\leq$ in "Hasse diagram".

Examples:

1. For $n \in \mathbb{N}$ let $P = \{1, 2, \ldots, n\}$ with the usual order $\leq$.

2. Let $B_n$ be the set of subsets of $\{1, 2, \ldots, n\}$ with $S \leq T$ if $S \subseteq T$.

3. Let $Q_n$ be the set of chains of $\{1, 2, \ldots, n\}$ with $S \leq T$ if $S \subseteq T$.

4. Let $T_n$ be the set of partitions of $\{1, 2, \ldots, n\}$ with $\pi \leq \sigma$ if every block of $\pi$ is a subset of a block of $\sigma$.

Some defi.:

- interval $[u, t] = \{v \in P : s \leq v \leq t\}$
- if there is a minimum elt., call it $\hat{0}$
- if there is a maximum elt., call it $\hat{1}$
- $P$ is locally finite if all its intervals are finite
- subposet:
  - induced: $Q \subseteq P$ where, for $x, y \in Q$, $x \leq y$ implies $x \leq y$
  - hereditary: $Q \subseteq P$ and $x \leq y$ implies $x \leq y$

$P \nleq Q$ if there is a bijection $\varphi : P \to Q$ with $x \leq y$ if and only if $\varphi(x) \leq \varphi(y)$.

- chain: $x_1 < x_2 < \ldots < x_k$
- antichain: $\{x_1, \ldots, x_k\}$ with $x_i \nleq x_j$ for all $i, j$

- rank of $P$: size of longest chain
- $P$ is ranked if for any $x \leq y$, all maximal chains from $x$ to $y$ have the same length. "It has levels."

- $P \wedge Q$: poset on $P \cup Q$ with order $\leq$ from $P, Q$

- $P \vDash$ poset on $P \times \hat{0}$ with $(x, y) \leq (x', y')$ if $x \leq x'$ and $x \leq y'$.

$\hat{0} \leq \hat{1}$ and

$\hat{0} \nleq \hat{1}$.

$\hat{0}$ is a bottom and $\hat{1}$ is a top.

Similarly, $\vDash$ has a unique order ideal.