Another important source of such formulas:

Let

\[ 0 \rightarrow V_n \xrightarrow{\partial_n} V_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} V_0 \xrightarrow{\partial_1} W \rightarrow 0 \]

be an exact sequence of fin-dim vector spaces, that is, \( \text{im} \partial_i = \ker \partial_{i+1} \). Then

\[ \dim W = \sum_{i=0}^{n} (-1)^i \dim V_i \]

Proof: Induction on \( n \).

More generally, consider the "binomial determinant"

\[ \det \left( \begin{array}{c} a_i \\ b_j \end{array} \right)_{1 \leq i, j \leq n} \]

for \( 0 \leq a_1 < \ldots < a_n \) integers
\( 0 \leq b_1 < \ldots < b_n \)

These determinants appeared in algebraic geometry (Habous - Classes de Chern d'un produit tensoriel)

Geisel-Viennot: why are they positive?

Lindstrom-Geisel-Viennot Lemma: (Korlin - McGregor)

Let \( G \) be a directed graph with no directed cycles.

Let \( s_1, \ldots, s_n \) be "sources"
\( t_1, \ldots, t_m \) be "sinks."

Assume all ratings (in vertex-disjoint paths) from \( \{s_1, \ldots, s_n\} = S \)

to \( \{t_1, \ldots, t_m\} = T \) connect \( s_i \) to \( t_j \). \( s_i \) \( t_j \). Then

\[ \det \left( a_{ij} \right)_{1 \leq i, j \leq n} = \# \text{ of ratings from } S \rightarrow T. \]

- There is a version allowing cycles
- There is a version with edge weights: \( a_{ij} = \sum_{P \text{ path } i \rightarrow j} w(P) \)
- \( \det(a_{ij}) = \sum_{P \text{ paths}} w(P) \)
- If other points are possible, \( \det(a_{ij}) = \sum_{P \text{ paths}} \sigma_{ij}(P) w(P) \)