Note: If \( G_1 = (V_1, E_1), G_2 = (V_2, E_2) \) then \( G_1 \times G_2 \) is the graph on \( V_1 \cdot V_2 \) with edge:

\[
(a_1, b_1) \in E_1 \Rightarrow (a_1, b_2) \in E_1 \times G_2
\]

\[
(a_2, b_2) \in E_2 \Rightarrow (a_1, b_2) \in G_1 \times E_2
\]

**Note:** \( G_n = (\cdot)^n \)

**Prop.** If \( L(G_1) \) has eigenvalues \( \lambda_1, \ldots, \lambda_a \)

\( L(G_2) \) has eigenvalues \( \mu_1, \ldots, \mu_b \)

then \( L(G_1 \times G_2) \) has eigenvalues \( \lambda_i + \mu_j \) \( 1 \leq i \leq a \)

\( 1 \leq j \leq b \)

**Pf.** Take eigenvectors \( \mathbf{r} \) of \( L(G_1) \) with eigenval \( \lambda \)

\( \mathbf{s} \) of \( L(G_2) \) with eigenval \( \mu \)

Let \( \mathbf{t} = (\mathbf{r}_i \cdot \mathbf{s}_j) \) \( 1 \leq i \leq a \)

\( 1 \leq j \leq b \)

\[ \mathbf{L}(\mathbf{L}) \mathbf{r} = \lambda \mathbf{r} \]

\[ \mathbf{L}(\mathbf{L}) \mathbf{s} = \mu \mathbf{s} \]

The entry \( (i, j) \) of \( L(G_1 \times G_2) \mathbf{t} \) is:

\[
(\deg v_i + \deg w_j) r_i s_j - \sum_j r_j s_j = \lambda_i r_i s_j
\]

\[
: = (\deg v_i, \text{deg } w_j)
\]

The characteristic polynomial of the Laplacian is:

\[
\det (\mathbf{L} - \lambda \mathbf{I}) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_a)
\]

The root of \( -\lambda \) is the sum of the roots of the

\( n \)th order polynomial, which are equal

\( \lambda, \ldots, \lambda_n \)
Matrix-tree Theorem (directed version)

Let \( D \) be a directed graph on \( [n] \). ("digraph")

The Laplacian of \( D \) is

\[
L_{ij} = \begin{cases} 
-(\# \text{ edges } i \rightarrow j) & i \neq j \\
\text{outdegree}(i) - (\# \text{ loops at } i) & i = j 
\end{cases}
\]

A directed spanning tree rooted at \( v \) is one

where all edges point toward \( v \).

\[
\text{(# directed spanning tree}) = \det (\text{adj} \text{ppal} \text{graph}) \\
\text{rooted at } v \\
= \frac{1}{n} \lambda_1 \cdots \lambda_{n-1}
\]

Corollary: This is independent of \( v \! \).
With a bit of advanced planning to make sure we cover the whole graph, we get a stronger result.

**Theorem:** Let $D$ be an Eulerian graph on $|V|$.
Consider an edge $e = v \rightarrow w$.

- $E(D,v) = \#$ oriented spanning trees of $D$ rooted at $v$.
- $E(D,e) = \#$ Eulerian walks of $D$
  starting at $e$.

Then

$$E(D,e) = E(D,v) \frac{(\text{outdeg}(v) - 1)!}{\text{indeg}(v)!}$$

**Proof:**
Let $T$ be one such tree.
Each vertex $i$ has a unique edge on $T$
pointing towards $v$. Linearly order the other
outdeg($i$)-1 arbitrarily.
(For $i=v$, order the outedges $\neq e$).

We claim this gives an Eulerian walk $E$ by:

- Start with $e$
- Every time you enter a vertex $i$, leave using the lowest unused out-edge.
  (If none available, use the out-edge in $T$).

**Ex:**

$$E = E_D = \frac{5!}{2!} \div 2$$

# of Eulerian walks = $n^{n-2} \cdot [(n-2)!]^n$