Since \( \prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n \)

\( \prod_{n=1}^{\infty} (1-x^n) = 1-x-x^2+x^5+x^3-x^{12}-x^{15}+... \)

When we multiply these and compute coefficients of \( x^n \) we get

\( 0 = p(n) - p(n-1) - p(n-3) + p(n-4) - p(n-11) + ... \)

This recurrence is the best way to compute \( p(n) \), but there are other ways of computing \( p(n) \) only. Also, \( p(n) \sim \frac{n^{1/2}}{4\sqrt{3}} n \)

(Compare with \( n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \))

Prop: The \# of self-conjugate partitions of \( n \) equals the \# of partitions of \( n \) into odd parts.

PE

\[ \begin{array}{c}
6 + 6 + 5 + 3 + 3 + 2 & \rightarrow & 11 + 9 + 5
\end{array} \]

Now that we've played enough with formal power series to know what we might need to worry about, let's discuss why we don't need to worry.

Let \( R = \text{com. ring}, \ (\text{for us usually } R = \mathbb{R} \ or \ \mathbb{C}) \)

A formal power series is a sequence \( (a_0, a_1, a_2, ...) \) which we write \( \sum a_k x^k \) \((a_i \in R)\)

Write \( a_n = [x^n] A(x) \)

The ring of formal power series \( R[[x]] \) has operations:

\[ \begin{align*}
+ & : (a_n)_{n \in \mathbb{N}} + (b_n)_{n \in \mathbb{N}} = (a_n + b_n)_{n \in \mathbb{N}} \\
\cdot & : (a_n)_{n \in \mathbb{N}} \cdot (b_n)_{n \in \mathbb{N}} = (a_0 b_0 + a_1 b_{-1} + ... + a_n b_{-n})_{n \in \mathbb{N}} \\
\end{align*} \]

(consistent with our power series notation)

We have \( 0 = (0,0,...) \), \( 1 = (1,0,0,...) \)

Basic: assoc. of \( + \), of \( \cdot \)

comm. of \( + \), of \( \cdot \)

dist. of \( + \), of \( \cdot \)

See: EC1, Sec 1.1
Ian Niven, "Formal Power Series" (Amer Math Monthly)
There is a distinction between formal and analytic power series, but:

**Principle:** Any identity of power series which holds analytically for small enough $l$, makes sense for formal power series, also holds in the ring of formal power series.

This is clearer through some examples:

**Ex. 1.** $\sum_{n=0} r^n x^n = \frac{1}{1-rx}$

Here this means $(\sum_{n=0} r^n x^n)(1-rx) = 1$, or a formal power series. But

$$[x^n] \text{ LHS} = \begin{cases} r^n + r^{n+1}(-r) = 0 & n \geq 1 \\ 1 & n = 0 \end{cases} \checkmark$$

**Ex. 2.** $\sum_{n=0} x^n / n! \sum_{n=0} (-x)^n / n! = 1$

This makes sense in $\mathbb{R}[[x]]$; i.e.

$$\sum_{n=0} \frac{1}{k!} (-1)^k = \begin{cases} 0 & n \geq 1 \\ 1 & n = 0 \end{cases}$$

This follows from $(1+1)^n = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases}$

An. We can also invoke analysis; this says $e^x e^{-x} = 1$

which is true for all $x \in \mathbb{C}$. Then just use:

Thus: if two power series represent the same function in a neighborhood of 0, then their coefficients are equal.

**Ex 3. A non-example**

The analytic identity $e^x + e^{-x}$ does not give an identity in $\mathbb{R}[[x]]$, because

$$\sum_{n=0} (x+1)^n/n!$$

is not a formal power series.

The series of $x^n$ has infinitely many contributions.

To make sense of (some) infinite sums, we need to define convergence in $\mathbb{R}[[x]]$.

Say $F_1(x), F_2(x), \ldots \rightarrow F(x)$

if for any $n$, there exists $N$ such that

$$[x^n] F_n(x) = [x^n] F_{n+1}(x) = \cdots = [x^n] F(x).$$

Let $\deg F(x) = \min n$ s.t. $[x^n] F(x) \neq 0$

**Prop.** $\sum_{j=0}^\infty A_j(x)$ converges $\iff \lim_{j \to \infty} \deg A_j(x) = \infty$.

So: $\sum_{n=0} \frac{(x+1)^n}{n!}$ does not, $\sum_{n=0} \frac{[x^n] F(x)}{n!}$ does.
So: If \( F(x), G(x) \in R[[x]] \), we can define

\[
F(G(x)) = \sum_{n=0}^{\infty} f_n \left( \sum_{m=0}^{\infty} g_m x^m \right)^n \quad \text{iff} \quad g_0 = G(0) = 0.
\]

Prop: \( \prod_{j=0}^{\infty} (1 + A_j(x)) \) converges \( \iff \) \( \lim_{n \to \infty} \deg A_j(x) = 0 \)
\( \iff \) \( A_j(0) = 0 \)

Ex. 4 We showed

\[
\sum_{n=0}^{\infty} P_{\leq k}(n)x^n = \prod_{n=0}^{\infty} \frac{1}{1-x^n}
\]

And concluded

\[
\sum_{n=0}^{\infty} P(n)x^n = \prod_{n=0}^{\infty} \frac{1}{1-x^n} = \prod_{n=0}^{\infty} (1+x^n+x^{2n}+\ldots)
\]

The RHS is defined and what we are doing is taking the limit of (1) as \( k \to \infty \); the terms stabilize to that of (2).

Prop: \( R[[x]] \) is an integral domain
if \( R \) is an integral domain

PF: If \( A(x)B(x) = 0 \) when \( A(x) = \sum_{n=0}^{\infty} a_n x^n \)
\( B(x) = \sum_{n=0}^{\infty} b_n x^n \)

Then \( A(x)B(x) = \sum_{n=0}^{\infty} a_n b_n x^n + \ldots \)

Ex. 5 The Catalan GF satisfies
\[
(1-2x+C(x))^2 = 1-4x
\]
Now:
\[
(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n
\]

satisfies
\[
(1-4x)^{1/2} = 1-4x
\]

since it satisfies \( f_0 = 0 \) for \( \|x\| < 1/4 \). Then
\[
[-2xC(x)]^2 = [(1-4x)^{1/2}]^2
\]

Now, \( A = B \Rightarrow (A-B)(A+B) = 0 \Rightarrow A = \pm B \).

Since they both have \( [Ex.1, \ldots, 1-2xC(x) = (1-4x)^{1/2} \]

"Calculus": Define \( \left( \sum_{n=0}^{\infty} f_n x^n \right)' = \sum_{n=0}^{\infty} (n+1)f_{n+1} x^n \)

We have

\[
(FG)' = F'G + FG' \quad (F(0))' = F'G(x)G(x) \quad (G(0) = 0)
\]

So, e.g. if \( G'(x) = F'(x)/F(x) \quad G(0) = 0 \quad F(0) = 1 \)
\( G'(x) = \frac{\log F(x)}{x} \quad G(x) = \log F(x) \rightarrow F(x) = e^{G(x)} \)