1. (Dull sequences) A sequence of positive integers is dull if for any \( k > 1 \) which appears in the sequence, the number \( k > 1 \) appears at least once before the first occurrence of \( k \).

(a) Find \( a_n(m) \), the number of dull sequences of length \( n \) where the largest number is \( m \). We show our desired number by first seeing the correspondence dull sequences have with set partitions.

The constraint we are given tells us that if the number \( k \) appears, we must have that \( 1, 2, ..., k - 1 \) must also appear to the left of \( k \)'s first appearance. For a given dull sequence of length \( n \) with a largest number \( m \), we show a 1-1 correspondence to a \( m \)-set partition of \([n]\). Let \( \sigma = \sigma_1...\sigma_n \) be a dull sequence of length \( n \) with largest number \( m \). Then we know \( \sigma \) contains the numbers \( 1, ..., m \). To get its corresponding set partition \( A_1 \sqcup ... \sqcup A_m \) we define the sets by

\[
A_i = \{ j : \sigma_j = i \}
\]

For example consider the dull sequence \( \sigma = 1123223114444255 \). Since the length of \( \sigma \) is 17 it must correspond to a set partition of \([17]\). By definition of \( A_i \) we get that partition

\[
\{1, 2, 3, 9, 10\} \sqcup \{4, 6, 7, 15\} \sqcup \{5, 8\} \sqcup \{11, 12, 13, 14\} \sqcup \{16, 17\}
\]

Thus we see that for each set \( A_i \) its minimum value corresponds to the first appearance of \( i \). Therefore given any set partition of \([n]\), order the sets \( A_i \) in increasing order by its minimum values. Do the reverse to what we did to sets \( A_i \) to create a sequence. By construction this will have to be a dull sequence.

Hence we have that

\[ a_n(m) = S(n, m) \]

where \( S(n, k) \) is the Stirling number of the second kind.

2. (Another generating function for Stirling numbers) In class we considered the (ordinary and exponential) generating functions for Stirling numbers \( S(n, k) \) when \( k \) is fixed and \( n \) varies. Now we consider the one where \( n \) is fixed and \( k \) varies: \( p_n(x) = \sum_{k=1}^{n} S(n, k)x^k \).

(a) Prove that \( p_0(x), p_1(x), p_2(x), ... \) satisfies \( p_n(x + y) = \sum_{m=0}^{n} \binom{n}{m} p_m(x)p_{n-m}(y) \).

We begin by seeing that

\[
p_m(x)p_{n-m}(y) = \left( \sum_{k=1}^{m} S(m, k)x^k \right) \left( \sum_{k=1}^{n-m} S(n-m, k)x^k \right)
\]

\[
\sum_{k=2}^{n} \sum_{j=1}^{k-1} \binom{n}{k} \binom{m}{j} S(m, j)S(n-m, k-j)x^iy^{k-j}
\]