1. Exercise 1.

1.1. Dull sequences must be the same as set partitions. Let \( A(m + n, m) \) denote the set of dull sequences of length \( n + m \geq 1 \) and maximum \( m \geq 1 \) and let \( A(n + m, m) \) be its cardinality. Define the combinatorial class \( A(m + \cdot, m) \) that contains the sequences in \( A(m + n, m) \) for any \( n \geq 0 \) along with the size function \(|\cdot| = \text{“length of the sequence”}\).

Thanks to the definition of dull sequences it is possible to split any element in the class \( A(m + \cdot, m) \) into some “irreducible” dull sub-sequences as follows:

\[
A_1 \underbrace{a_{11} \ldots a_{1\alpha_1}}_{\text{sub-sequence}} A_2 \underbrace{a_{21} \ldots a_{2\alpha_2}}_{\text{sub-sequence}} \ldots A_m \underbrace{ma_{m1} \ldots a_{m\alpha_m}}_{\text{sub-sequence}}
\]

where each of the sub-sequences \( A_i \) only has elements in \([i]\) and its first element is \( i \). If we define the sub class \( A_i \) as the set of all sequences \( A_i \) that satisfy the above property, then we clearly have that

\[
A_i(z) = \sum_{n=1}^{\infty} i^{n-1} z^n = \frac{z}{1 - iz},
\]

since for each \( A_i = ia_1 \ldots a_{1n} \) the position \( a_{ij} \) can be any element of \([i]\). Hence, we have that

\[
A(m + \cdot, m)(z) = A_1(z) \cdot A_2(z) \cdot \ldots \cdot A_m(z) = \frac{z}{1 - z} \frac{z}{1 - 2z} \ldots \frac{z}{1 - mz},
\]

but in Lecture 12 we saw that the RHS of the above equation was the GF of the Stirling Numbers, and hence we must have that \( A(m + n, m) = S(n + m, m) \).

1.2. Proving that \( A(n, m) = S(n, m) \). Now that he have seen what the numbers \( A(n, m) \) must be, let us prove the aforementioned equality. Let \( a_1 \ldots a_n \) be a dull sequence in \( A(n, m) \). We have two possibilities for \( a_n \): (1) \( a_n \) is the first occurrence of the maximum \( m \), therefore \( a_1 \ldots a_{n-1} \in A(n-1, m-1) \). (2) \( a_n \) is not the first occurrence of the maximum \( m \), hence \( a_1 \ldots a_{n-1} \in A(n-1, m) \). Since \( a_n \) could had been any element of \([m]\) we must have the equality

\[
A(n, m) = A(n - 1, m - 1) + mA(n - 1, m).
\]