3. Let's prove a bijection $f$ of the Dyck paths of length $2n$ that start in the origin to finite binary trees on $[2n]$. Let $T$ be a finite binary tree on $[2n]$ and assign to the root of $T$ the point $(0,0)$. Then performing the Depth-First Search assign each of the points of the Dyck path, that is:

- If you move through an edge going to the left for the first time, go up in the Dyck path.
- If you move through an edge going to the right for the first time, go down in the Dyck path.

This clearly leads to a bijection, because if you have a path $P$ then let $f((0,0))$ be the root. Every time you go up in P right after the point $(a,b)$ put a right son to $f((a,b))$ (namely $f((a+1,b+1))$). Every time you go down in P right after the point $(a,b)$ put a left son to the first $f((c,d))$ without a left son. This evidently gives a tree.

Now let's see that for a Dyck path $P$ $a(P)$ (the number of steps up before the first step up) is equal to the number of times you can go down taking only to the left from the root of $f(P)$, and that $b(P)$ (the number of times that $P$ returns to the $x$-axis) is equal to the number of times you can go down taking only to the right from the root of $f(P)$. This follows by construction: The maximum number of times you can go down to the left of the root is the number of times you can go up in the path before you take a step down which is $a(P)$; For the second proof we proceed by strong induction on the length of a Dyck Path: For $n = 1$ where the length of the path is $2n$ trivial. Now suppose by induction that for every $j < n$ the result is true. Take a Dyck path $P$ of length $2n$ suppose $b(P) = k$, divide $P$ into subDyck paths $P_1, \ldots, P_k$ where each $P_i$ begins at the $i-1$th intersection with the $x$-axis and ends in the $i$th intersection with the $x$-axis. Because each $P_i$ has length $2j < 2n$ with $j < n$, by induction $b((P_i)) = 1$ is the number of times you can go down taking only the right from the root of $f(P_i)$, $f(P)$ can be reconstructed by placing the root of each $P_{i+1}$ at the rightest corner from the root. Hence $b(P) = b(P_1) + \ldots + b(P_k)$ is the number of times you can go down taking only the right from the root of $f(P)$. By the induction principle it follows.

If you take a reflection over the vertical line traced across the root of $f(P)$ for a Dyck path $P$, you are interchanging the left walks from the root to the leftest corner only taking the left with the right walks from the root to the rightest corner only taking the right. Hence the problem follows.

(I worked with Felipe Vargas and Angela María Castañeda)

4. Let $x,y,q$ be elements of a non-commutative ring such that $yx = qxy$, and $q$ commutes with $x$ and $y$. Consider $(x+y)^n$, let $i \in [n]$ lets find the coefficient $A$ of $x^i y^{n-i}$. Consider the multiset $M = \{x^i y^{n-i}\}$, we
know that in order to get how many times appears $x'y^{n-i}$ we have to consider all the possible permutations of $M$, and, by knowing that $q$ commutes with $xy$, reorder each permutation to get $x'y^{n-i}$. Let $\pi \in S_M$, suppose that $x < y$ so that every time that $y$ appears before an $x$ it will be recorded as an inversion of $\pi$. Because $q$ commutes with $xy$, whenever we set $yx = qxy$ we will get a $q$. Hence $q$ will appear $\text{inv}(\pi)$ times, that is the coefficient of each $x'y^{n-i}$ is set by:

$$A = \sum_{\pi \in S_M} q^{\text{inv}(\pi)} \binom{n}{i, n-i} q^{(n)}$$

Since this happens for each $i$ we have:

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} q^i y^{n-i}$$

Which is what we wanted to prove.

Let $x_1, \ldots, x_m, q$ be elements of a non-commutative ring such that $x_ix_i = qx_ix_j$ for $i < j$ and $x_iq = qx_i$ for all $i$. Consider $(x_1 + \ldots + x_m)^n$, let $\alpha_1, \ldots, \alpha_m$ be a $m$-composition of $n$ lets find the coefficient $A$ of $x_1^{\alpha_1} \ldots x_m^{\alpha_m}$. Consider the multiset $M = \{x_1^{\alpha_1} \ldots x_m^{\alpha_m}\}$, we know that in order to get how many times appears $x_1^{\alpha_1} \ldots x_m^{\alpha_m}$ we have to consider all the permutations of $M$ and, by knowing that $q$ commutes as in the hypothesis, reorder each permutation to get $x_1^{\alpha_1} \ldots x_m^{\alpha_m}$. Let $\pi \in S_M$, suppose that $x_i < x_j$ for all $i < j$ so that every time that $x_j$ appears before an $x_i$ it will be recorded as an inversion of $\pi$. Because $q$ commutes as stated in the hypothesis, whenever we set $x_ix_i = qx_ix_j$ for $i < j$ we will get a $q$. Hence $q$ will appear $\text{inv}(\pi)$ is set by:

$$A = \sum_{\pi \in S_M} q^{\text{inv}(\pi)} \binom{n}{\alpha_1, \ldots, \alpha_m}$$

Since this happens for each $m$-composition of $n$ we have

$$(x_1 + \ldots + x_m)^n = \sum_{\alpha_1}^{n} \ldots + \sum_{\alpha_m}^{n} q^{(n)} x_1^{\alpha_1} \ldots x_m^{\alpha_m}$$

Which is what we wanted to prove.

(1 worked with Nicolas Peña)

5. Let find the generating functions of both the partitions whose elements appear at least twice, and the partitions whose elements are not congruent with 1 mod 6.

First let find the generating function of the partitions whose elements appear at least twice. Its clear that the generating function for a partition of $n$ is

$$\prod_{k=1}^{\infty}(1 + x^k + x^{2k} + \ldots) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Now we want the partitions with at least 1 repetition for each element. That is,