and if we place number $i$ below number $k$, and $k + 1$ is on the left of $k$, then we place the same number $i$ below $k + 1$, otherwise we place $i + 1$ below $k + 1$

$$
\begin{array}{cccccccccc}
4 & 6 & 3 & 7 & 7 & 2 & 8 & 5 & 1 \\
& & & 1 & 1 & & & & \\
4 & 6 & 3 & 7 & 7 & 2 & 8 & 5 & 1 \\
& & & 1 & 1 & & & & \\
4 & 6 & 3 & 7 & 7 & 2 & 8 & 5 & 1 \\
1 & 1 & 1 & & & & & & \\
4 & 6 & 3 & 7 & 7 & 2 & 8 & 5 & 1 \\
1 & 1 & 1 & 1 & 2 & 1 & & & & \\
\end{array}
$$

And keep doing this to get

$$
\begin{array}{cccccccccc}
4 & 6 & 3 & 7 & 7 & 2 & 8 & 5 & 1 \\
1 & 2 & 1 & 3 & 3 & 1 & 4 & 2 & 1 \\
\end{array}
$$

This is a bijection because of the fact that in a full sequence, the first occurrence of $k - 1$ is before the last occurrence of $k$.

6. (Bonus problem: cycles of even and odd permutations.)

(a) Let $e_n$ be the total number of cycles among all even permutations of $[n]$, and $o_n$ be the total number of cycles among all odd permutations of $[n]$. Prove that

$$
e_n - o_n = (-1)^n(n - 2)!
$$

We have the polynomial equality

$$
x(x + 1) \cdots (x + n - 1) = \sum_{k=1}^{n} C(n, k)x^k
$$

where $C(n, k)$ is the number of permutations of $[n]$ with $k$ cycles, and we skip term $k = 0$ because $C(n, 0) = 0$.

Now let's take the derivative of both sides,

$$
\sum_{i=0}^{n-1} x(x + 1) \cdots (x + i) \cdots (x + n - 1) = \sum_{k=1}^{n} kC(n, k)x^{k-1}
$$

where $(x + i)$ means we skip that term.

If we plug the value $x = -1$, the RHS is closely related to what we want, because when $n$ is even, the parity of the number of cycles of a permutation is the same as the parity of the permutation itself, and if $n$ is odd, is the opposite.

When we plug $-1$ in the polynomial almost all terms of the LHS are cero, except the one we skip the factor $(x + 1)$, so we get

$$
-(n - 2)! = \sum_{k=1}^{n} kC(n, k)(-1)^{k-1}
$$
\[
= \sum_{k, \text{ odd}} kC(n, k) - \sum_{k, \text{ even}} kC(n, k)
\]

\(kC(n, k)\) is \(k\) times the number of permutations with \(k\) cycles, so it counts the total number of cycles of the permutations with \(k\) cycles, so because of what we said earlier,

if \(n\) is even \(\sum_{k, \text{ odd}} kC(n, k) = o_n\) and \(\sum_{k, \text{ even}} kC(n, k) = e_n,\)

then \(-(n - 2)! = o_n - e_n \Rightarrow e_n - o_n = (n - 2)!\)

and if \(n\) is odd \(\sum_{k, \text{ odd}} kC(n, k) = e_n\) and \(\sum_{k, \text{ even}} kC(n, k) = o_n,\)

then \(-(n - 2)! = e_n - o_n,\) and in both cases we get what we wanted.

(b) Give a bijective proof of (a).

We know \(f : S_n \rightarrow S_n : w \mapsto w(12)\) the composition with the transposition (12) is a bijection between even and odd permutations, but let’s see more detailed other bijections defined by this mapping. A permutation having 1 and 2 in the same cycle is mapped in one having them in different cycles, and the converse is true, too.

Let’s say \(ed_n =\) number of even permutations with 1 and 2 in different cycles, \(es_n =\) number of even permutations with 1 and 2 in the same cycle, and \(od_n, os_n\) defined similarly. Let \(Ced_n =\) total number of cycles of even permutations with 1 and 2 in different cycles, and the others be defined similarly.

With the mapping \(f\) we have bijections that shows the following equalities

\[ed_n + es_n = od_n + os_n\]

\[ed_n = os_n\]

\[es_n = od_n\]

And let’s change the problem in terms of the new definitions.
\(e_n = Ced_n + Ces_n\) and \(o_n = Cod_n + Cos_n\)

If we have a permutation with 1 and 2 in different cycles and we map it through \(f\) the cycles of 1 and 2 form a new cycle, so the number of cycles is decreased by one. Then

\[Ced_n = Cos_n + ed_n\]

similarly

\[Ces_n = Cod_n + es_n\]

So we obtain \(e_n - o_n = ed_n - es_n\) meaning that what we want is equal to the number of even permutations with 1 and 2 in different cycles minus the number of even permutations with 1 and 2 in the same cycle. Now let’s prove \(ed_n - es_n = (-1)^n(n - 2)!\) by induction.
If \( n = 2 \) there is only one even permutation, \((1)(2)\) and have 1 and 2 in different cycles, so \( ed_2 = 1 \) and \( es_2 = 0 \), so it is true for \( n = 2 \). Now suppose \( ed_{n-1} - es_{n-1} = (-1)^{n-1}(n-3)! \)

We have the following recurrence relation, \( ed_n = ed_{n-1} + od_{n-1}(n - 1) \) depending if \( n \) is the only one in its cycle or not, if it is, then the parity of the permutation doesn’t change and we have \( ed_{n-1} \) possibilities for the other \( n - 1 \), and if \( n \) is not the only one in its cycle, then when we add it to another cycle the parity of the permutation changes, so we have \( od_{n-1} \) possibilities for the \( n - 1 \) when we erase \( n \) and to put it again we have \( n - 1 \) possibilities for choosing the image of \( n \).

\[
ed_n = ed_{n-1} + od_{n-1}(n - 1) = ed_{n-1} + es_{n-1}(n - 1)
\]

the last change because of the equalities we had at the beginning. Similarly we obtain

\[
es_n = es_{n-1} + os_{n-1}(n - 1) = es_{n-1} + ed_{n-1}(n - 1)
\]

With this we get

\[
ed_n - es_n = ed_{n-1} + es_{n-1}(n - 1) - es_{n-1} - ed_{n-1}(n - 1)
= (ed_{n-1} - es_{n-1})(1 - (n - 1)) = (-1)^{n-1}(n - 3)!(2 - n) = (-1)^n(n - 2)!
\]