we see that for \( m \) 0’s we have \( m + 1 \) places to put our runs. As we can only place one run in a space the number of ways to do this is \( \binom{m+1}{k} \) and so the combined result is
\[
\binom{n-1}{k-1} \binom{m+1}{k}.
\]

(3) (Sequences of subsets)

(a) Let \( k, n \geq 1 \) be given. Find the number of sequences \( S_0, S_1, \ldots, S_k \) of subsets of \([n]\) such that for any \( 1 \leq n \leq k \) we have either:
\[
S_i \supseteq S_{i-1} \quad \text{and} \quad |S_i - S_{i-1}| = 1,
\]
or
\[
S_i \subseteq S_{i-1} \quad \text{and} \quad |S_{i-1} - S_i| = 1.
\]
First, as there are \( 2^n \) possible subsets there are \( 2^n \) possibilities for the set \( S_0 \). Now, with every transition from a subset \( S_i \) to \( S_{i+1} \) we “toggle” the membership of one of its elements. This means that for the transitions we are simply counting the number of sequences of length \( k \) that use numbers from \([n]\). Since there are \( n^k \) such sequences the answer is
\[
2^n n^k.
\]

(b) Prove that there are exactly
\[
\frac{1}{2^n} \sum_{i=0}^{n} \left( \binom{n}{i} (n-2i)^k \right)
\]
such sequences with the additional property that \( S_0 = S_k = \emptyset \).
For ease in notation, let
\[
T_{k,n} = \frac{1}{2^n} \sum_{i=0}^{n} \left( \binom{n}{i} (n-2i)^k \right).
\]
Since every element that is “toggled” in must also be “toggled” out, we are simply counting the number of sequences of length \( k \) of numbers from \([n]\) that use each number an even number of times—for example, we can have
\[
1 \ 2 \ 3 \ 2 \ 3 \ 1 \ 4 \ 4 \ 1 \ 4 \ 4 \ 1.
\]
We will call such a sequence a strongly even.

Now, if \( k \) is odd, this will never work, so the number of ways is 0, which agrees with this formula. Now assume that \( k \) is even. If \( n = 1 \), then there is 1 possibility (all 1s), so
\[
T_{k,n} = 1
\]
for all even \( k \). Now, assume that the formula works for some fixed value \( n \). To count the number of strongly even sequences that use the numbers in \([n+1]\), we break it down into the cases where element \( n + 1 \) appears \( i \) times, and count the number of possibilities for the other \( n \) numbers in the remaining \( k - i \) positions in the sequence:

(i) “\( n + 1 \)” appears 0 times, so the remaining numbers appear \( k \) times. There are \( T_{k,n} \) ways of doing this.

(ii) “\( n + 1 \)” appears 1 time. This is an odd number, and so there should be \( T_{k-1,n} = 0 \) ways of doing this.

(iii) “\( n + 1 \)” appears 2 times, so the remaining numbers appear \( k - 2 \) times. There are \( \binom{k}{2} \) ways for placing the numbers “\( n + 1 \)” and \( T_{k-2,n} \) ways of choosing the remaining numbers, so there are \( \binom{k}{2} T_{k-2,n} \) ways.

(iv) In general, if “\( n + 1 \)” appears \( j \) times, there are \( \binom{k}{j} T_{k-j,n} \) ways. Take note that this works even for odd \( j \) as \( k - j \) is odd, so that while we are counting
\[
\sum_{j=0}^{k/2} \binom{k}{2j} T_{k-2j,n}
\]
it is much more convenient to use the equivalent sum
\[
\sum_{j=0}^{k} \binom{k}{j} T_{k-j,n}
\]
despite the fact that half of its summands are zero.
So now to show that the number of strongly even sequences using the set \([n + 1]\) is equal to \(T_{n+1,k}\) we use some algebra:

\[
\sum_{j=0}^{k} \binom{k}{j} T_{k-j,n} = \sum_{j=0}^{k} \binom{k}{j} \left[ \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (n-2i)^{k-j} \right] = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \left[ \sum_{j=0}^{k} \binom{k}{j} (n-2i)^{k-j} \right] = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (n + 1 - 2i)^{k} = \frac{1}{2^{n+1}} \cdot 2 \left[ \binom{n}{0} (n+1)^{k} + \binom{n}{1} (n-1)^{k} + \binom{n}{2} (n-3)^{k} + \cdots + \frac{n}{n} (n-1)^{k} \right] = \frac{1}{2^{n+1}} \left[ \binom{n}{0} (n+1)^{k} + \binom{n}{0} (n-1)^{k} + \binom{n}{1} (n-1)^{k} + \binom{n}{n} (n-1)^{k} \right] = \frac{1}{2^{n+1}} \left[ \binom{n+1}{0} (n+1)^{k} + \binom{n+1}{1} (n-1)^{k} + \cdots + \frac{n+1}{n} (n-1)^{k} + \frac{n}{n} (n-1)^{k} \right] = \frac{1}{2^{n+1}} \sum_{i=0}^{n} \binom{n+1}{i} (n + 1 - 2i)^{k} = T_{k,n+1}.
\]

And the induction is complete.

(4) (Permutations fixed by \(\hat{\cdot}\)) Let \(\hat{\cdot} : S_n \rightarrow S_n\) be the fundamental transformation of \(S_n\). Prove that the number of permutations \(w\) in \(S_n\) such that \(\hat{w} = w\) is the Fibonacci number \(f_{n+1}\).

We will show that there exists a bijection between compositions of \(n\) consisting of only 1s and 2s and permutations that are fixed by \(\hat{\cdot}\). The bijection \(\phi\) is best given by this example where \(n = 12\):

\[
\phi(1 + 2 + 1 + 1 + 2 + 2 + 1 + 2) = (1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) = (1) (32) (4) (5) (76) (98) (10) (1211), \text{ standard form}
\]

\[
= \left( \begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
1 & 3 & 2 & 4 & 5 & 7 & 6 & 9 & 8 & 10 & 12 & 11
\end{array} \right)
\]

The function \(\phi\) creates
- 1-cycles \((i)\) at position \(i\) (according to the two-line format), and
- 2-cycles \((i + 1 i)\) at positions \(i\) and \(i + 1\), switching the two numbers in the two-line format.

To best see how this works, rewrite the composition as partial sums, so that

\[
\begin{align*}
a_1 &= 1 \\
a_2 &= 3 \\
a_3 &= 4 \\
a_4 &= 5 \\
a_5 &= 7 \\
a_6 &= 9 \\
a_7 &= 10 \\
a_8 &= 12.
\end{align*}
\]

These numbers are strictly increasing and are the initial numbers of the cycles (or the records) and if a number \(i\) is skipped, then it is in the cycle with \(i + 1\). This function is injective as different compositions