1a Give a combinatorial proof that for any positive integers \( n \geq k \),
\[
\binom{n}{k} = \frac{n(n-1)}{k(k-1)}
\]

**Solution.** A car lot has \( n > 0 \) different black cars. How many different ways are there to put \( k \leq n \) of the cars into a garage so that one of the \( k \) cars in the garage is also painted red?

We could choose \( k \) cars out of \( n \) to move to the garage in \( \binom{n}{k} \) ways. Then there are \( k \) choices for which to paint one of them red. Thus we can accomplish the task in \( k \binom{n}{k} \) ways.

Instead we could first choose which car to paint red and move into the garage; this can be done in \( n \) ways. Then there are \( n-1 \) cars left and we need to choose \( k-1 \) of them to move into the garage with the red car. So we can accomplish the task in \( n \binom{n-1}{k-1} \) ways.

Since we were counting the same thing,
\[
k \binom{n}{k} = n \binom{n-1}{k-1}.
\]

Dividing by \( k \),
\[
\binom{n}{k} = \frac{n(n-1)}{k(k-1)}.
\]

1b Give a combinatorial proof that for any positive integers \( n \geq k \),
\[
\sum_{l=k}^{n} \binom{l}{k} = \binom{n+1}{k+1}
\]

**Solution.** Let’s count the number of \((k+1)\)-element subsets of \([n+1] = \{1,2,...,n+1\}\).

This can be counted straightforwardly as \( \binom{n+1}{k+1} \).
On the other hand, we can count the total number of subsets with the desired property by counting how many we have with a certain largest element and adding them together. Each of the \((k + 1)\)-element subsets of \([n + 1] = \{1, 2, ..., n + 1\}\) will have a largest element between \(k + 1\) and \(n + 1\) (inclusive). Once we determine what the largest element \(l\) of the set is, we must choose \(k\) additional elements which are less than the largest element to make the \((k + 1)\)-element subset. There are \(l - 1\) elements smaller than \(l\) to choose from. Thus the number of \((k + 1)\)-element subsets of \([n + 1] = \{1, 2, ..., n + 1\}\) is

\[
\sum_{l=k+1}^{n+1} \binom{l-1}{k} = \sum_{l=k}^{n} \binom{l}{k}.
\]

Since we’re counting the same objects,

\[
\sum_{l=k}^{n} \binom{l}{k} = \binom{n+1}{k+1}.
\]

2 A binary word is a word consisting of 0s and 1s. A run is a maximal string of consecutive 1s. For example the word 11010111011 has 4 runs. Find the number of binary words having exactly \(m\) 0s, \(n\) 1s, and \(k\) runs.

**Solution.** First we must break the \(n\) 1s up into \(k\) runs. This is equivalent to counting the number of \(k\)-compositions of \(n\), so it can be done in \(\binom{n-1}{k-1}\) ways. There must be a 0 between each of the \(k\) runs; thus the position of \(k - 1\) 0s is forced. We are left with \(m - (k - 1) = m - k + 1\) 0s. There are \(k + 1\) places to put these 0s which would result in different binary words: before the first run, between two of the runs, or after the last run. Thus there are \(\binom{k+1}{m-k+1}\) ways to place the remaining 0s. Since these are independent tasks, by the multiplication principle we know that there are \(\binom{n-1}{k-1} \cdot \binom{k+1}{m-k+1}\) binary words having exactly \(m\) 0s, \(n\) 1s, and \(k\) runs.

3a Let \(k, n \geq 1\) be given. Find the number of sequences \(S_0, S_1, ..., S_k\) of subsets of \([n]\) such that for any \(1 \leq i \leq k\) we have either \(S_i \supset S_{i-1}\) and \(|S_i - S_{i-1}| = 1\), or \(S_i \subset S_{i-1}\) and \(|S_{i-1} - S_i| = 1\).

**Solution.** Consider a subset \(S_{i-1}\) of \([n]\) with \(p\) elements. How many possibilities do we have for \(S_i\)? If \(S_i \supset S_{i-1}\) and \(|S_i - S_{i-1}| = 1\), it means we have added one element to \(S_{i-1}\) to make \(S_i\). There are \(n - p\) ways to do this. If, instead, \(S_i \subset S_{i-1}\) and \(|S_{i-1} - S_i| = 1\), we have deleted one element from \(S_{i-1}\) to achieve \(S_i\). There are \(p\) ways to do this. Thus in general there are \((n-p) + p = n\) possibilities for \(S_i\) given \(S_{i-1}\). There are \(2^n\) choices for which subset of \([n]\) is \(S_0\). Since the sequence \(S_0, S_1, ..., S_k\) has \(k\) additional subsets, and each transition from one set to another is independent, we have a total of \(2^nn^k\) sequences with the desired property.