Our next goal: **CLASSIFYING FINITE COXETER GROUPS**

(W, S) Coxeter gp
S = \{s_1, \ldots, s_n\}
M_{ij} \rightarrow Coxeter
matric

- Basis of V: \( \Delta = \{a_1, \ldots, a_n\} \)
- Bilinear form
  \( \langle d_i, d_j \rangle = -\cos (\pi / m_{ij}) \)

Review of bilinear form

Representing matrix: Let \( e_1, \ldots, e_n \) be a basis.
Suppose that \( \mathbf{u} = \sum_i u_i e_i \), \( \mathbf{v} = \sum_j v_j e_j \), then

\[
\langle \mathbf{u}, \mathbf{v} \rangle = \sum_i u_i v_j \langle e_i, e_j \rangle
\]

\[
= [u_1 \ldots u_n] [\langle e_i, e_j \rangle]_{ij} [v_1 \ldots v_n] = \mathbf{u}^T \mathbf{E} \mathbf{v}
\]

So in coordinates, \( \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{E} \mathbf{v} \)

If \( f_1, \ldots, f_n \) is a different basis, say

\[
\begin{bmatrix}
  f_1 & f_2 \\
  f_n & f_n
\end{bmatrix} = \begin{bmatrix}
  e_1 & e_2 \\
  e_1 & e_n
\end{bmatrix} M
\]

then \( \mathbf{u} = [f_1 \ldots f_n] \mathbf{u} = [e_1 \ldots e_n] M^{-1} \mathbf{u} \)

So \( u \) has coord. \( M^{-1} \mathbf{u} \) wrt \( \{f_1, \ldots, f_n\} \)

Also check:
\( F = \langle f_i, f_j \rangle \) \( \Rightarrow M^T [\langle e_i, e_j \rangle] M = \mathbf{E} \)

so now
\[
\langle \mathbf{u}, \mathbf{v} \rangle = (M^{-1} \mathbf{u})^T \mathbf{E} M (M^{-1} \mathbf{v}) = \mathbf{u}^T \mathbf{E} \mathbf{v}
\]

Now representing matrix.

If \( \langle \mathbf{x}, \mathbf{x} \rangle \) is symmetric, then \( F \) is symmetric so it can be diagonalized:

\( M^{-1} \mathbf{E} M = \mathbf{D} \) for \( M \) orthogonal

\( M^{-1} = M^T \)

A bilinear form \( \langle \mathbf{x}, \mathbf{x} \rangle \) is positive definite if \( \langle \mathbf{x}, \mathbf{x} \rangle > 0 \) for all \( \mathbf{x} \neq 0 \).

Claim: A symmetric \( \langle \mathbf{x}, \mathbf{x} \rangle \) is positive definite.

It is the Euclidean inner product wrt some basis.

\( \uparrow \) Clear \( \downarrow \): diagonalize it \( \Rightarrow \) get \( \mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \)

so this corresponds to an orthogonal basis.
Theorem: \( W \) is finite if \( \langle , \rangle \) is positive definite.

\( \uparrow \): (Assume some topology for a bit.)

Note \( W \subset \text{GL}(V) \).

\[ \text{NH invertible matrices} = \mathbb{R}^{n^2} \]

Claim: \( W \) is discrete.

(No \( u \in W \) such that every \( \text{nbhd} \) of \( u \) contains only many points of \( W \)).

pf. Remember \( W \) acts on \( T \) by cone.

(one chamber per \( w \in W \))

\[ \text{Take } x \in D \]

\[ \text{Let } U = \{ A \in \text{GL}(V) \mid Ax \in wD \} \]

The only point of \( W \) here is \( w \).

Now if \( \langle , \rangle \) is positive definite then it is Euclidean, so \( W \) acts by real reflections, so \( W \subset O(V) = \text{orthogonal} \) gp.

\[ \text{NH matrices with } A^{-1} = A^T \]

Now \( O(V) \subset \mathbb{R}^{n^2} \) is:

- closed because \( O(V) \) is cut out by algebraic equation,

- banded because orthogonal matrices have orthonormal columns, so they have length 1.

So \( O(V) \) is compact.

If \( W \subset O(V) \) was infinite, it would have an accumulation point.

For the other direction we need a bit of representation theory.

**Def**: A representation of a group \( G \) is a homomorphism \( p: G \rightarrow \text{GL}(V) \) for some vector space \( V \).

(A way of seeing \( G \) as a group of invertible linear transforms (or invertible matrices once we choose a basis for \( V \)).)