Our next goal: "Coxeter groups are automatic"

\[ \text{reduced words} \xrightarrow{\text{walks in a finite graph}} \]

**Ex:** \( \infty \)

\[ \text{reduced words: } \emptyset, a, ba, b, ab, aba, bab, abab, \ldots \]

This will take some work:

root pair \( \rightarrow \) small roots \( \rightarrow \) automatically

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**Root Poset**

Recall \( W \)-Coxeter gp

\[ S = \{ s_1, \ldots, s_n \} \text{ gens} \]

\[ \Delta = \{ \alpha_1, \ldots, \alpha_n \} \text{ simple roots (basis for } V) \]

\[ \langle \alpha_i, \alpha_j \rangle = -\cos \left( \frac{\pi}{m_{ij}} \right) \]

\[ \Phi = \{ w \alpha_i \mid w \in W, 1 \leq i \leq n \} = \text{roots} \]

\[ \Phi^+ = \Phi^* \cup \Phi^- \]

positive root \( w \alpha \)

reflection \( w \sigma \)

\[ \langle \ell(w) \alpha, \ell(w) \alpha \rangle > 0 \]

\[ \ell(w \alpha) > \ell(w) \]

\[ W \text{ sends } l(w) \text{ roots from } \Phi^+ \text{ to } \Phi^- \]
Def. The depth of a root $\beta > 0$ is the minimum length of a walk such that $w\beta < 0$.

Prop. $dp(\beta) = \frac{1}{2}(l(\ell(\beta)) + 1)$

If $\beta = w_0 x_i$, then $\ell(w) \geq \ell(w_i) + 1$ if $w = w_0 x_i w^{-1}$.

Then
\[\frac{m}{2}(w x_i) < 0 \quad \Rightarrow \quad dp(\beta) \leq \ell(w) + 1 = \frac{1}{2}(\ell(\ell(\beta)) + 1)\]

For $\beta = s_0 s_b \ldots s_z \beta < 0$

Then $s_0$ sends $s_b \ldots s_z \beta < 0$ to $> 0$.

So $d_a = s_b \ldots s_z \beta$

$s_0 = s_b \ldots s_z t s_2 \ldots s_b$

$t = s_z \ldots s_b s_0 s_b \ldots s_z \rightarrow \ell(t) \leq 2 dp(\beta) + 1$

Def. The root pair on $S_n^+$ is defined by $\beta < \gamma \iff \gamma = s^\beta$ for $s \in S$.

Note
- graded by depth
- edges labelled by $S$.

The root pair of $S_4$ is $\{a_1, a_2, a_3, a_1 a_2 a_3\}$.

(Parenthesis: The number of antichains in $\mathcal{C}(n-1) = \frac{1}{n} \binom{2(n-1)}{n}$

the Catalan number

(answer to $> 100$ problems? Not yet, but several already-

this is an active area of research.)

Root pair for $A_4$

$\langle i, j \rangle = i a_j + j a_i + k a_i a_j$

- $\langle i, a_j \rangle = -1/2$, $\langle a_i, a_j \rangle = 1$
- $s_1 a_i = -a_1$
- $s_2 a_2 = a_1 + a_2$
- etc.
- $s_3 a_3 = a_1 + a_3$.

\[
\begin{array}{ccc}
1 & 12 & 21 \hline
12 & 2 & 21 \hline
21 & 11 & 10 \hline
11 & 100 & 01 \hline
100 & 010 & 001
\end{array}
\]