Let $(W,S)$ be a Coxeter system.

Let $w = s_1 \ldots s_r \in W$ be a reduced word.

Prove that the roots $w$ sends to the $-ve$ set are precisely the $r$ roots $\beta_i = s_r s_{r-1} \ldots s_i (a_i)$, i.e.

\[ \beta_1 = s_r s_{r-1} \ldots s_2 (a_i) \]
\[ \beta_2 = s_r s_{r-1} \ldots s_3 (a_i) \]
\[ \vdots \]
\[ \beta_{r-1} = s_r (a_{r-1}) \]
\[ \beta_r = a_r. \]

**Proof:**

$w = s_1 \ldots s_r \in W$ is reduced $\Rightarrow l(w) = r$.

By prop 4.4.4 ($B \& R$), proven in class, the # of

the roots which $w$ sends is exactly $r$.

So, we need only to show that $\sum \frac{\beta_i}{\beta_i}$

$w \beta_i < 0, \quad i = 1, \ldots, r$

$\beta_i$ as specified above and that they're all distinct.

Note $H \cdot \beta_i = (s_{i-1} \ldots s_r) s_r s_{r-1} \ldots s_{i+1} (a_i)$

$$= s_{i-1} \ldots s_i (a_i)$$

Since $s_i \ldots s_r$ is reduced, all $s_i \ldots s_{i+t}$ (prefixes) are also reduced.

--- end
Consequently,
\[ \lambda((s_1 \cdots s_i) s_i) = \lambda(s_1 \cdots s_{i-1}) = \lambda s_i = \lambda(s_1 \cdots s_i) \]
and
\[ \lambda((s_r \cdots s_{i+1}) s_i) = r - i + 1 > r - i = \lambda(s_r \cdots s_{i+1}). \]

Thus, by prop 4.2.5 (ii) B & B,
\[ w(\beta_i) = (s_i \cdots s_i) \alpha_i < 0 \]
\[ \beta_i = (s_r \cdots s_{i+1}) \alpha_i > 0, \quad i = 1, \ldots, r. \]

Now we prove \( \beta_i \) are all distinct.

Assume not: The let \( j > i \) but \( \beta_j = \beta_i \).

Then \( s_r \cdots s_{j+m}(\alpha_j) = s_r \cdots s_{i+m}(\alpha_i) \)

which implies \( \alpha_j = s_j \cdots s_{i+m}(\alpha_i) \).

So,
\[ -\alpha_j = s_j \cdots s_{i+1}(\alpha_i). \]

In the case \( j = i+1 \), \( -\alpha_j = \alpha_i \).

In all cases, since \( s_j \cdots s_{i+1} \alpha_i \geq s_j \cdots s_{i+m} \alpha_i \)

the RHS of \( \Box \) is \( > 0 \) but the LHS of \( \Box \) is \( < 0 \)

since all simple roots are \( > 0 \), which \( \Rightarrow \) \( \Box \) is \( \Leftrightarrow \).

\[ \therefore (\Rightarrow) \quad Q.E.D. \]

In general