Repeating this process several times, either we are done, or we get that

\[
\underbrace{s'_{1}s_{1}s'_{1}s_{1}\cdots}_{k \text{ times}} = \underbrace{s_{1}s'_{1}s_{1}s'_{1}\cdots}_{k \text{ times}}
\] (3)

and it is enough to show that

\[
\underbrace{s'_{1}s_{1}s'_{1}s_{1}\cdots}_{k \text{ times}} \sim \underbrace{s_{1}s'_{1}s_{1}s'_{1}\cdots}_{k \text{ times}}
\] (4)

But equation (3) implies that \(k\) is a multiple of \(m(s_{1}, s'_{1})\), since we know the order of \(s_{1}s'_{1}\) is \(m(s_{1}, s'_{1})\). Therefore, we can divide \(s'_{1}s_{1}s'_{1}s_{1}\cdots\) into blocks of size \(m(s_{1}, s'_{1})\), and replace each block to conclude that (4) holds.

6. Note that

\[
283457619 = 283457619
\]
\[
\rightarrow 283457691
\]
\[
\rightarrow 283459671
\]
\[
\rightarrow 293458671
\]
\[
\rightarrow 298453671
\]
\[
\rightarrow 298543671
\]
\[
\rightarrow 298563471,
\]

so \(283457619 \leq 298563471\).

7. Let \(M = \{\dim(E_{i} \cap F_{j})\}_{1 \leq i, j \leq n}\), and let \(j \in [n]\). The numbers in the \(j\)-th column of \(M\) correspond to the sequence

\[
\dim(E_{1} \cap F_{j}), \dim(E_{2} \cap F_{j}), \ldots, \dim(E_{n} \cap F_{j}),
\]

which is clearly a non-decreasing sequence. Note that the kernel of the canonical homomorphism

\[
f : E_{i} \cap F_{j} \to E_{i}/E_{i-1}
\]
is \(\ker(f) = (E_{i} \cap F_{j}) \cap E_{i-1} = E_{i-1} \cap F_{j}\), so it induces an injection

\[
\hat{f} : (E_{i} \cap F_{j})/(E_{i-1} \cap F_{j}) \hookrightarrow E_{i}/E_{i-1}.
\]

Therefore

\[
\dim(E_{i} \cap F_{j}) - \dim(E_{i-1} \cap F_{j}) = \dim((E_{i} \cap F_{j})/(E_{i-1} \cap F_{j})) \leq \dim(E_{i}/E_{i-1}) = 1,
\]

which shows that the sequence of numbers in the \(j\)-th column of \(M\) increases at most 1 at each step. Define

\[
S_{j} = \{i \in [n] : \dim(E_{i} \cap F_{j}) - \dim(E_{i-1} \cap F_{j}) = 1\},
\]

that is, \(S_{j}\) is the set of steps where the sequence of numbers in the \(j\)-th column increases. Since \(\dim(E_{0} \cap F_{j}) = \dim(0) = 0\) and \(\dim(E_{n} \cap F_{j}) = \dim(F_{j}) = j\) then the set \(S_{j}\) contains exactly \(j\) elements. Moreover, note that if \(j > 1\) and \(i \notin S_{j}\) then

\[
\dim(E_{i} \cap F_{j}) - \dim(E_{i-1} \cap F_{j}) = 0
\]
and so $E_i \cap F_j = E_{i-1} \cap F_j$, thus

$$E_i \cap F_{j-1} = (E_i \cap F_j) \cap F_{j-1} = (E_{i-1} \cap F_j) \cap F_{j-1} = E_{i-1} \cap F_{j-1}$$

and

$$\dim(E_i \cap F_{j-1}) - \dim(E_{i-1} \cap F_{j-1}) = 0,$$

showing that $i \notin S_{j-1}$. We have then that $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n$. Since for all $j$ we have that $\#S_j = j$, then for all $j$ there is a unique number $\sigma(j) \in S_j \setminus S_{j-1}$ (where we take $S_0 = \emptyset$). Note that $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\} = S_n = [n]$, so it is clear that $\sigma$ is a permutation of the set $[n]$. Moreover, $M$ is the rank table of the permutation $\sigma$, because for any pair $i, j$ we have that the number of dots in the northwest corner above the point $(i, j)$ in the rank table of $\sigma$ is

$$\# \{ k \leq j : \sigma(k) \leq i \} = \# \{ k \leq j : S_k \setminus S_{k-1} \leq i \}
= \# ( (S_1 \setminus S_0) \cup (S_2 \setminus S_1) \cup \cdots \cup (S_j \setminus S_{j-1}) ) \cap [i])
= \# (S_j \cap [i])
= M_{i,j}$$

directly from the definition of $S_j$. 