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Math 490  
HW 2

#1

Fix a triangle $T$ in the infinite equilateral triangular grid and let reflection through its bounding hyperplanes be denoted by $s_1, s_2, s_3$ respectively. Then $s_i^2 = e$ for $i = 1, 2, 3$. Also $(s_is_j)^3 = e$ for $i \neq j$ since such a pair generate a group isomorphic to the dihedral group of the hexagon in which this relation holds.

To see that $S = \{s_1, s_2, s_3\}$ generate all symmetries it suffices to show that $S$ generates the twelve triangles adjacent to $T$ in the grid. These are indeed given by $s_i (i = 1, 2, 3)$, $s_is_j (1 \leq i \neq j \leq 3)$, and $s_1s_2s_1, s_1s_3s_1, s_2s_3s_2$. After reflecting the hyperplanes appropriately, we may now consider any one of these triangles as our fixed triangle and we are done by induction.

Finally, let $w = t_1 \ldots t_k$ be a reduced word in the group generated by $S$ and let $s \in S$ such that $l(sw) \leq l(w)$. We show that $sw = t_1 \ldots \hat{t}_i \ldots t_k$ for some $i$. If $s = t_1$, then $sw = \hat{t}_1 t_2 \ldots t_k$. Otherwise, $s = t_2$ and $t_3 = t_1$ (by the fact that $w$ was taken reduced). So $st_1 t_2 t_3 \ldots t_k = st_1 t_1 t_4 \ldots t_k = t_1 t_4 \ldots t_k$. But this is exactly $t_1 t_2 \hat{t}_3 \ldots t_k$ and thus we are done.

#2

For a Coxeter system $(W,S)$ let $l : W \rightarrow \mathbb{N}$ take $w \rightarrow k$ where $w = s_1 \ldots s_k$ is a reduced expression for $w$. Then

(a) $l(uv) \equiv l(u) + l(v) \pmod{2}$.

Proof. Let $u = s_1 \ldots s_k$ and $v = s_{k+1} \ldots s_n$ be reduced expressions for $u$ and $v$. If $uv = s_1s_2 \ldots s_n$ is a reduced expression for $uv$, then $l(uv) = l(u) + l(v)$ and the result follows. On the other hand, if $uv = s_1s_2 \ldots s_n$ is not reduced, then we repeatedly apply the deletion property to obtain a reduced expression for $uv$. Each application of the deletion property preserves the parity of $n$ (since it removes two letters at a time), so $l(uv) \equiv l(u) + l(v) \pmod{2}$.

(b) $l(w^{-1}) = l(w)$

Proof. Since $l(ww^{-1}) = l(e) = 0$, we have that $l(w) \equiv l(w^{-1}) \pmod{2}$, by part (a). Suppose $w = s_1 \ldots s_k$ and $w^{-1} = t_1 \ldots s_{k-2m}$ are reduced expressions for $w$ and $w^{-1}$ (where $m \in \mathbb{Z}_{\geq 0}$). We show that $m = 0$. Since $e = ww^{-1} = s_1 \ldots s_k t_1 \ldots t_{k-2m}$ has length zero, we apply deletion repeatedly to the RHS. But each application must remove an $s_i$ and a $t_j$ since our original expressions for $w$ and $w^{-1}$ were reduced. Therefore, to obtain the empty word we must apply deletion precisely $k$ times. Thus, $m = 0$ as desired.

(c) $l(sw) = l(w) \pm 1$
Proof. If \( w = s_1 \ldots s_k \) be a reduced expression for \( w \). If \( sw \) is reduced, then clearly \( l(sw) = l(w) + 1 \). If \( sw \) is not reduced, then \( sw = s_1 \ldots \hat{s}_i \ldots s_k \) for some \( i \) by the exchange property. This is a reduced expression for \( sw \) since otherwise \( w = s(sw) \) would be expressible as a word of length less than \( k \) which contradicts \( w = s_1 \ldots s_k \) being reduced.

(d) The distance function \( d(u,v) = l(uv^{-1}) \) is a metric on \( W \).

Proof. Since no word can have negative length, the distance function is non-negative. Suppose \( d(u,v) = 0 \). Then \( l(uv^{-1}) = 0 \) implies that \( u = (v^{-1})^{-1} = v \). Thus \( d \) is positive-definite. Since \( vu^{-1} = (uv^{-1})^{-1} \), we have \( l(uv^{-1}) = l(vu^{-1}) \) by part (b), and hence that \( d \) is symmetric. Finally, if \( s_1 \ldots s_k \) and \( t_1 \ldots t_j \) are reduced expressions for \( uw^{-1} \) and \( vw^{-1} \) respectively, then we have a (not necessarily reduced) expression \( uv = uw^{-1}uv^{-1} = s_1 \ldots s_k t_1 \ldots t_j \) of length \( l(uw^{-1}) + l(vw^{-1}) \). So \( d(u,v) \leq d(u,w) + d(w,v) \). So \( d \) is a metric on \( W \).

#3

\[
l(uv) = l(u) + l(v) \text{ iff there is no } t = wsw^{-1} \in T \text{ such that } l(ut) < l(u) \text{ and } l(tv) < l(v).
\]

Proof.

Let \( u = s_1 \ldots s_k \) and \( v = t_1 \ldots t_l \) be reduced expressions for \( u, v \), respectively.

For the forward direction, suppose there is a \( t = wsw^{-1} \in T \) such that \( l(ut) < l(u) \) and \( l(tv) < l(v) \). Then by Corollary 1.14.4 we have \( ut = s_1 \ldots \hat{s}_i \ldots s_k \) for some \( i \) and \( tv = t_1 \ldots \hat{t}_j \ldots t_l \) for some \( j \). Thus

\[
l(uv) = l(uttv)
\]

\[
\leq k + l - 2
\]

\[
< k + l
\]

\[
= l(u) + l(v).
\]

So \( l(uv) \neq l(u) + l(v) \).

For the other direction, we show the \( l(uv) < l(u) + l(v) \) implies the existence of a \( t \in T \) such that \( l(ut) < l(u) \) and \( l(tv) < l(v) \). Toward this end, note \( l(uv) < l(u) + l(v) \) implies that \( uv \) is not reduced. So \( uv = s_1 \ldots \hat{s}_i \ldots s_k t_1 \ldots \hat{t}_j \ldots t_l \) since we started with reduced expressions for \( u \) and \( v \). By Corollary 1.14.4 there is a \( t \in T \) such that \( tv = t_1 \ldots \hat{t}_j \ldots t_l \). Thus \( l(tv) < l(v) \) and (again, by Corollary