**Are two ideals equal? (An application)**

To test whether \( I = J \), choose any monomial order \( \prec \) and compute the reduced Gröbner basis \( G \) and \( H \) for \( I \) and \( J \). Then

\[ I = J \iff G = H \] (by thrm abov)

\[ G = H \iff \langle G \rangle = \langle H \rangle = J \]

**Ex. 2**

\[ I = \langle x^3y - xy^2 + 1, x^2y^2 - y^3 - 1 \rangle \]

\[ J = \langle xy^3 + y^3 + 1, x^3y - x^3 + 1, x + y \rangle \]

For \( \prec = \text{lex} \) with \( x > y \),

\[ G = H = \{ x + y, y^4 - y^3 + 1 \} \]

So \( I = J \).

**Ex. 1**  
lex., \( x > y \) for:

\[ I = \langle x^2 + xy + y^2 + y, xy - xy + y^2, xy - y^2 \rangle \]

\[ h_1, h_2, h_3 \]

\[ S(h_1h_2) = S(h_1h_3) = 0 \mod \{ h_1, h_2, h_3 \} \]

\[ S(h_2h_3) = y^5 - y^2 \mod \{ h_1, h_2, h_3 \} \]

\[ h_4 \]

\[ S(h_1h_4) = S(h_2h_4) = S(h_3h_4) = 0 \mod \{ h_1, h_2, h_3, h_4 \} \]

So \( x^2 + xy + y^2 \) is a Gröbner basis. Then

\[ \{ x^2 + xy + y^2, y^5 - y^2 \} \]

is a minimal Gröbner basis. Now

\[ x^2 + xy + y^4 \equiv x^2 + xy + y^4 \mod y^5 - y^2 \]

So

\[ \{ x^2 + xy + y^4, y^5 - y^2 \} \text{ is the reduced Gröbner basis.} \]

Elimination Theory:
(Solving Systems of Polynomial Equations)
(An application)

**Ex.**

\[ \begin{align*}
x^2 + 2xy + y^2 - 2x - 2y &= 0 \\
x^2 + y^2 &= 1
\end{align*} \]

Elliptic Circle

Clearer manipulation:

\[ 5y^4 - 4y^3 = 0 \]

\[ y = 0 \quad \text{or} \quad y = \frac{4}{5} \]

\[ \downarrow \quad \downarrow \]

\[ x = 1 \quad x = -\frac{3}{5} \]

How to do this in general?

**Same idea:**

1. Look for \( p(x_n) = 0 \), solve for \( x_n \).
2. Look for \( q(x_{n-1}, x_n) = 0 \), solve for \( x_{n-1} \) for each sol. in 1.
3. Look for \( r(x_{n-2}, x_{n-1}, x_n) = 0 \), solve for \( x_{n-2} \) for each sol. in 2. in 1.
This "amount" to computing the elimination ideals

\[ I_i = I \cap \mathbb{F}[x_1, \ldots, x_n] \]

**Theorem** Let \( G = \{ g_1, \ldots, g_m \} \) be a G. b. for \( I \) wrt the lex order \( x_1 > \cdots > x_n \), and let \n
\[ G_i = G \cap \mathbb{F}[x_1, \ldots, x_n] \]

Then \( G_i \) is a G. b. for \( I_i \) \( (wrt \lex, x_1 > \cdots > x_n) \).

So simple!
In particular, \( I_i \neq 0 \iff G_i \neq \emptyset \)

\[ \text{[desirable for elimination!]} \]

** Pf.** Need: \( \text{in} (I_i) = \langle \text{in} (G_i) \rangle \)

Let \( f \in I_i \). Since \( G \) is a G. b.,

\[ \text{in} (f) = a_1 \text{in} (g_1) + \cdots + a_m \text{in} (g_m) \]

\[ \text{in} (f) = \alpha_1 \text{in} (g_1) + \cdots + \alpha_m \text{in} (g_m) \]

\[ \text{in} (f) \in \langle \text{in} (g_1), \ldots, \text{in} (g_m) \rangle \] as an ideal in \( \mathbb{F}[x_1, \ldots, x_n] \)

But in lex order, if \( \text{in} (g_a) \) involves only \( x_{i_1} \ldots x_{i_k} \)
then \( g_a \) involves only \( x_{i_1} \ldots x_{i_k} \)

\[ \text{so} \ g_a \in G_i \]

So \( \text{in} (f) \in \langle \text{in} (G_i) \rangle \)

**Computing \( INJ \):**

(An application)

If \( I = \langle f_1, \ldots, f_a \rangle \) and \( J = \langle g_1, \ldots, g_b \rangle \) then

\[ I + J = \langle f_1, \ldots, f_a, g_1, \ldots, g_b \rangle \]

\[ IJ = \langle f_1 g_1, \ldots, f_1 g_b, \ldots, f_a g_1, \ldots, f_a g_b \rangle \]

\[ INJ = ? \]

**Prop.** a) \( t I + (1-t) J \) is an ideal in \( \mathbb{F}[t, x_1, \ldots, x_n] \)

b) \( \text{in} \cap J = (t I + (1-t) J) \cap \mathbb{F}[x_1, \ldots, x_n] \)

So \( \text{INJ} \) is the first elim. ideal of \( t I + (1-t) J \), wrt. \( t > x_1 > \cdots > x_n \), and we can compute it!

** Pf.** a) clear
b) \( \subseteq \) clear

2: Let \( f = tf_1 + (1-t)f_2 \)

\[ f \in \mathbb{F}[x_1, \ldots, x_n], f_1 \in I, f_2 \in J \]

Plugging in \( t = 0 \), we get \( f = f_2 \)

\[ \Rightarrow f = f_1 \]

\[ \Rightarrow f \in \text{INJ} \]