Idea of Buchberger: Can I generate missing initial monomials using combin. of $g_i$, ..., $g_m$?
It suffices to check the "basic" combination $S(g_i, g_j)$:

How do you construct a Gröbner basis?

Buchberger's Algorithm

Input: $<$, $I$, $F = \{f_1, ..., f_n\}$ generating $I$
Output: A Gröbner basis $G$ of $I$ (containing $F$)

Let $G := F$
Let $B := (F)$

While $B \neq 0$:

Pick \{fg\} $\in B$

Let $r = S(f, g) \mod G$

If $r \neq 0$ then

$G := G \cup \{r\}$

$B := B \setminus \{\text{fg}\}$

$B := B \setminus \{\text{fg}\}$

I.e.: Check all pairs \{fg\} in $G$:

- If $S(f, g) = 0 \mod G$, ok. Go to next pair.
- If $S(f, g) = r \neq 0 \mod G$, add $r$ to $S$

Repeat until all pairs are ok.

A Gröbner basis $\{g_1, ..., g_m\}$ is minimal if

- each in $g_i$ is monic
- no in $g_j$ is a multiple of in $g_i$ ($j \neq i$)

It is reduced if

- each in $g_i$ is monic
- no term of $g_j$ is a multiple of in $g_i$ ($j \neq i$)

Theorem: Given $I$, there is a unique reduced Gröbner basis.

Pf. Existence: Start with any Gröbner basis.

1. make each in $g_i$ monic
2. remove any unnecessary in $g_i$
3. divide each $g_i$ by $g_1, ..., g_i, ..., g_k$

and let the remainder be $r_i$.

In $g_i$ is not a multiple of in $g_j$ ($i \neq j$).
So it also occurs in $r_i \Rightarrow \text{in}(g_i) = \text{in}(r_i)$

So \{r_1, ..., r_k\} is a reduced Gröbner basis.

Uniqueness: Sup G = \{g_1, ..., g_m\} and G' = \{g_1', ..., g_m'\}

Any two minimal Gröbner bases $G$ and $G'$ I had the same size and leading terms, by HW2).

Say in $g_i$ = in $g_i'$ = $h_i$; let $f_i = g_i - g_i' \in I$.

in($f_i$) $\in \text{in}(I)$ $\Rightarrow$ some $\text{in}(g_j)$ $\mid$ in($f_i$)

$\Rightarrow$ $f_i = 0$ $\Rightarrow$ $g_i = g_i'$.