The 3-D case

\[ J = \langle x^4, y^4, z^4, x^3 y^2, x y^3 z^2, x^2 y^3 z^3 \rangle \]

Def. The Buchberger graph \( \text{Buch}(I) \) of a monomial ideal \( I = \langle m_1, \ldots, m_r \rangle \) has vertices \( i, \ldots, r \) and

\[(ij \text{ if and only if } \exists k \text{ with } \deg(m_k) > \deg(m_i) \text{ and } \deg(m_k) > \deg(m_j)) \Rightarrow m_k \mid \text{lcm}(m_i, m_j) \text{ for all } x_a \mid \text{lcm}(m_i, m_j) \]

Prop. If \( \text{I} \) is strongly generic in \( \mathbb{R}[x, y, z] \), then \( \text{Buch}(I) \) is a planar, connected graph, canonically embedded in the surface of \( I \).

Sketch key: \( \text{lcm}(m, m') \) is on the surface, so draw \( m \sim \text{lcm}(m, m') \).
What if \( J \) isn't strongly generic?
- Deform it as \( J_\epsilon \) with \( \epsilon \) shifted over by small deformations.
- Resolve \( J_\epsilon \) by a planar graph \( G_\epsilon \).
- Set \( \epsilon \) back to 0 to get a resol. of \( J \).

Ex: \( I = \langle x, y, z \rangle^3 = \langle x^3, x^2y, x^2z, xy, xz, \ldots \rangle \) in \( \mathbb{F}[x, y, z] \)
\[
I_\epsilon = \langle x^3, x^2y, x^2z, xy, xz, \ldots \rangle \text{ in } \mathbb{F}[x^{0.9}, y^{0.9}, z^{0.9}]
\]

---

**Minimal Free Resolution of \( J \):**
\[
0 \to R^7 \xrightarrow{\partial_6} R^{12} \xrightarrow{\partial_5} R^6 \xrightarrow{\partial_4} I \to 0
\]

**Hilb. series of \( J \):**
\[
\begin{align*}
\text{vertex labels:} & \quad \text{edge labels:} & \quad \text{face labels:} \\
(x^4 + x^2y^3) = (x^4 + x^2y^2 + xy^3) + (x^4 + x^2y^3) \\
(1-x)(1-y)(1-z)
\end{align*}
\]

**Lifting. Decomp. of \( J \):**
\[
J = \langle x, y, z \rangle \cap \cdots \cap \langle x^3, y^3, z^3 \rangle \quad \text{(face labels)}
\]