Q: Squarefree and Borel-fixed monomial ideals are nice, but what about the other monomial ideals?

**The 2-D case**

\[ I = \langle m_1, \ldots, m_r \rangle = \langle x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r} \rangle \]

Basic syzygies:

\[
\begin{align*}
\gamma(x^5y^2) - x^2(x^3y^3) &= 0 \\
\gamma(x^3y^3) - x^3(x^0y^4) &= 0
\end{align*}
\]

(The one between \(m_1,m_2\) is implied)

So the minimal free resolution of \(R/I\):

\[
0 \to \mathbb{Q}^2 \to \mathbb{Q}^3 \to \mathbb{Q}^r \to R \to R/I \to 0
\]

In general,

**Prop** The minimal free resol. of \(R/I\)

in \(k[x,y]=R\) has the form

\[
0 \to \mathbb{Q}^{r-1} \to \mathbb{Q}^r \to R \to R/I \to 0
\]

\[
\begin{align*}
\gamma_{1,0} &\quad \gamma_{0,1} \\
\gamma_{2,1} &\quad \gamma_{1,2} \\
\gamma_{3,2} &\quad \gamma_{2,3}
\end{align*}
\]

order canon.

inner canon.

**Cor**

\[
K(R/I; x,y) = 1 - \frac{1}{r} \sum_{i=0}^{r-1} x^{a_i}y^{b_i} + \frac{1}{r} \sum_{i=0}^{r-1} x^{a_i}y^{b_i}
\]

**Prop** I has independent indecomposable decomp

\[
I = \langle y^{b_1} \rangle \cap \langle x^{a_1}y^{b_2} \rangle \cap \langle x^{a_2}y^{b_3} \rangle \cap \ldots \cap \langle x^{a_r} \rangle
\]

**Rq** Draw picture.

Do these "resolution by picture" generalize?

Yes, but we need to go to higher dim.
Two general techniques

Say \( J = \langle x^4, y^4, z^4, x^2y^2z, x^2y^2z, x^2y^2z \rangle = \text{IF} \{x, y, z\} = \mathbb{R} \)

1. Reduce to squarefree case.

Let the "polarization" of \( J \) be

\( I = \langle x_1^2x_2^2x_3^2, y_1^2y_2^2y_3^2, z_1^2z_2^2z_3^2, x_1y_1z_1, x_2y_2z_2, x_3y_3z_3 \rangle, \ldots \rangle

\( \text{IF} \{x_1, y_2, \ldots, z_3, z_4\} = S \)

We know how to deal with this one.

Nice facts:

- \( R/J \cong (S/I) / \langle x_1-x_2, x_2-x_3, \ldots, z_3-z_4 \rangle \)

- From the min. free resol. / Hilbert series of \( S/I \)

We get that of \( R/J \) by setting \( x_i = x \)

\( y_i = y, z_i = z \)

Trouble

\( (\text{alg. of } I) \leftrightarrow (\text{comb./top of } \Delta) \)

\( \uparrow \text{huge!!} \)

In this case the free is

\( (1, 12, 56, 2294, 7833, 26246, 51) \)

So this is

- good for proving theorems
- bad for actually computing.

2. Reduce to Buell-fixed case

\( \downarrow \)

\( \text{gen} \text{ser. of } (J) = \langle x^4, x^2y, x^2z, \ldots \rangle \)

We know how to deal with this one.

Nice facts:

- Coarse Hilbert series are equal:

\( K(R/J; t) = K(R/\text{gen } J; t) \)

\( = 1 + 3t + 6t^2 + 10t^3 + 12t^4 + 12t^5 + \ldots \)

- Both numbers are bounded:

\( \beta_{i,a}(R/J) \leq \beta_{i,a}(R/\text{gen } J) \)

Trouble

Cannot we this to compute the fine Hilbert series or the min. free resolution.

Back to pictures: