Prop R Noetherian \iff R satisfies the ascending chain condition:
if \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \) are ideals then for some \( n \) we have \( I_n = I_{n+1} = I_{n+2} = \ldots \)

 Pf. See forum. \( \Box \)

This idea of "leading terms" gives us good generators. How do we do this in \( \mathbb{F}[x_1, \ldots, x_n] \)?

LT \( (2xy^2z - x^3 + 7x^2y^3) = ? \)

- lexicographically : \( x^2 \)
- degree, then lex: \( 7x^2y^2 \)

Lex Fix order, say \( x_1 > \ldots > x_n \). Say

\[
\mathbb{A}x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} > \mathbb{B}x_1^{b_1}x_2^{b_2} \cdots x_n^{b_n} \quad (a_1 > b_1, \ldots, \text{lex})
\]

if the first position where they differ has \( a_i > b_i \).

Grlex Fix order, say \( m_1 > m_2 \) if

- \( \deg m_1 > \deg m_2 \), or
- \( \deg m_1 = \deg m_2 \) and \( m_1 \text{ lex} > m_2 \).

Def A monomial order in \( \mathbb{F}[x_1, \ldots, x_n] \) (or \( \mathbb{Z}^n \)) is Noetherian

- if a total order on the set of monomials such that
- \( m \gg 1 \) for all \( m \)
- \( m \gg m_2 \) then \( mm_1 \gg mm_2 \) for all \( m \).

Check: - lex, grlex are monomial orderings.
- monomial orderings are well orderings (every non-empty set has a minimum element)
Def. Fix a monomial ordering $<$ on $\mathbb{F}[x_1, \ldots, x_n]$.

- If $\in \mathbb{F}[x_1, \ldots, x_n] \rightarrow \text{LT}(f) = \text{in}_<(f)$ is the leading term of $f$
- I ideal in $\mathbb{F}[x_1, \ldots, x_n] \rightarrow \text{LT}(I) = \text{in}_<(I) = \langle \text{in}_<(f) : f \in I \rangle$
- I ideal in $\mathbb{F}[x_1, \ldots, x_n] \Rightarrow \{g_1, \ldots, g_n\}$ is a Gröbner basis for $I$ if
  - $g_1, \ldots, g_n$ generate $I$
  - $\text{in}_<(g_1), \ldots, \text{in}_<(g_n)$ generate $\text{in}_<(I)$

Ex. $I = \langle x^2y - xy^2 + 1, x^2y^2 - y^3 + 1 \rangle \leq \text{lex with } x \succ y$

- $f_1 = x^2y$, $f_2 = x^2y^2$
- $\text{in}_<(f_1) = x^2y$, $\text{in}_<(f_2) = y^2$
- But $yf_1 - xf_2 = x - y \in I$, $\text{in}_<(x - y) = x$
- So $\{f_1, f_2\}$ is not a Gröbner basis. What is?

What do you use a Gröbner basis for?

Computation: Does $f \in I$?

- So $f_1 = \cdots = f_n = 0$.
- Find the relation between $f_1, \ldots, f_n$.

Theorems: Hilbert's basis theorem

Hilbert's syzygy theorem

Testing ideal membership: does $f \in I$?

To decide whether $f \in \langle g_1, \ldots, g_k \rangle$ we might use:

Division Algorithm
- $\text{in}_<(f) \geq \text{in}_<(g_i)$
- $r$ "minimal"

Goal: Write $f = g_1g_2 + \cdots + g_kg_k + r$ (has no monomial divisible by an in$(g_i)$)

Start with $g_1 = \cdots = g_k = r = 0$, and then

"peel off of $f$" by successively cancelling out the "largest" term:

(i) If $\text{in}_<(f) = m_1 \text{in}_<(g_1)$, for some $i$, let $f \mapsto f - m_1g_i$ (smaller in$_<$)

(ii) If not, let $f \mapsto f - \text{in}_<(f)$ (smaller in$_<$)

Repeat until $f \mapsto 0$

In the end we get $f = g_1g_2 + \cdots + g_kg_k + r$, no in$(g_i)$|in$(r)$

Ex. $f = x^2y + y$, $g_1 = xy + 1$, $g_2 = x + y$, $x \succ y$

$x^2y = \underbrace{(xy)}_{0} + \underbrace{(x + y)}_{0} + \underbrace{(x^2)}_{0}$

$x^2y = x(xy + 1) - 1(x + y) + 2y$

Different choices: $0(xy + 1) + (xy - y^2)(x + y) + (x^2y + y)$

The answer depends on a monomial ordering:

- $0$ choices in (i)