Theorem. The Hilbert series of the Stanley-Reisner ring $R/I_\Delta$ of $\Delta$ is:

$$H(R/I_\Delta; x) = \sum_{x=\Delta} \frac{\prod_{j=0}^{n} (1-x_j)}{\prod_{i=0}^{n} (1-x_i)}$$

Proof. A basis for $R/I_\Delta$ is given by the monomials not in $I_\Delta$:

$$x_{a_1}^{n_1}x_{a_2}^{n_2}...x_{a_n}^{n_n} \in I_\Delta \iff x_\tau | x_{a_1}^{n_1}x_{a_2}^{n_2}...x_{a_n}^{n_n} \quad \tau \in \Delta$$

So

$$H(R/I_\Delta; x) = \sum_{x=\Delta} \frac{x^\alpha}{\prod_{i=0}^{n} (1-x_i)}$$

$$= \sum_{\sigma \in \Delta} x^\alpha$$

$$= \sum_{\sigma \in \Delta} \frac{\prod_{j=0}^{n} (1-x_j)}{\prod_{i=0}^{n} (1-x_i)}$$

Example:

$$\Gamma = \begin{array}{c}
\text{a tetrahedron}
\end{array}$$

$$f_0 + f_1 + f_2 + f_3 + f_4$$

$$= 1 + f_0(1-t) + f_1(1-t)^2 + f_2(1-t)^3$$

Coerce:

$$3x^2 - 2x^2 - x^2 + 1 = \frac{2x^2 + 2x + 1}{(1-x)^3}$$

Def. The f-vector $(f_0, f_1, f_2, f_3)$ of $\Delta$ is:

$$f_i = \# \text{ of } i\text{-faces of } \Delta$$

The f-polynomial of $\Delta$ is:

$$\sum_{i=0}^{n} f_i t^i$$

The h-polynomials $h_0$ and h-vectors $(h_0, h_1, h_2, h_3)$ are:

$$\sum_{i=0}^{n} h_i t^i (1-t)^i = \frac{f(x)}{(1-t)^n}$$

Cor. With the coarse grading, if $\Delta$ is a simplicial complex, then:

$$H(R/I_\Delta; x) = h_\Delta(x)$$

Proof. $h_0(1) = f_0 > 0$.

Cor. $\dim(R/I_\Delta) = 1 + \dim \Delta$

Dehn-Sommerville Relations, if $\Delta$ is a simplicial complex, $\Delta$ is the boundary of a d-polytope, $h_\Delta$ is symmetric.
It turns out that free resolutions of square-free monomial ideals are very closely related to homology groups of simplicial complexes. So let's learn that.

**Algebraic topology:**

Top space $X \to \text{Alg. object } A(X)$.

So that if $X = \text{"y" then } A(X) = A(Y)$

**Ex. square = circle**

coffee cup = donut (homeomorphic)

$\circ \neq \circ$

$\circ \neq \circ$

By triangulating a surface, we make it a simplicial complex.

How do we detect the topology of a simplicial complex electrically?

E.g., how do we "find holes"?

Intuition: holes are cycles which aren't boundaries.

$\partial_t : C_t(\Delta) \to C_{t-1}(\Delta)$

$e_{a_1, \ldots, a_{k-1}} \mapsto \sum_{j=1}^{\binom{n}{j}} (-1)^{j} e_{a_1 \ldots \hat{a_j} \ldots a_{k-1}}$

This is the $i$-th boundary map.

**Prop.** $\partial_{t-1} \circ \partial_t = 0$

If $\partial_t : C_t(\Delta) \to C_{t-1}(\Delta)$

$e_{a_1, \ldots, a_{k-1}} \mapsto \sum_{j=1}^{\binom{n}{j}} (-1)^{j} e_{a_1, \ldots, \hat{a_j}, \ldots, a_{k-1}}$

$= \frac{\sum}{j} (-1)^{j} \left( \sum_{k<j} (-1)^{k} e_{a_1, \ldots, \hat{a_k}, \ldots, \hat{a_j}, \ldots, a_{k-1}} \right)$

$= \frac{\sum}{k>j} (-1)^{k} e_{a_1, \ldots, \hat{a_k}, \ldots, \hat{a_j}, \ldots, a_k}$

$= \sum_{r<s} e_{a_1, \ldots, \hat{a_k}, \ldots, \hat{a_j}, \ldots, a_k} (\sum (-1)^{r} e_{a_1, \ldots, \hat{a_k}, \ldots, \hat{a_j}, \ldots, a_k}) = 0$
So we have the (augmented/reduced) chain $\alpha$ of $\Delta$:

$$0 \to C_{d+1}(\Delta) \xrightarrow{\partial_{d+1}} \cdots \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) \to 0$$

Let $B_i(\Delta) = \text{Im} \partial_i$ be the boundary, and $Z_i(\Delta) = \text{Ker} \partial_i$ be the cycles.

And $\tilde{H}_i(\Delta) = \text{Ker} \partial_i / \text{Im} \partial_{i+1}$ is the $i$-th reduced homology group of $\Delta$.

**Ex.**

The boundary of the $i$-simplex (the sphere $S^{n-1}$)

$$\Delta = \partial \Delta = \{ e \in \mathbb{R}^n \mid e \neq [n] \} \quad n=3$$

$C_i(\Delta) = \text{span}_\mathbb{R} \{ e_1, e_2, \ldots, e_n \}$

which is almost like the free resolvent $\mathbb{R}[x_1, x_2]/(x_1, x_2)$ in HW2. That one gave an exact sequence:

$$0 \to R \xrightarrow{\partial_1} R \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_i} R \xrightarrow{\partial_{i+1}} 0$$

with basically the same map.

In the same way, $F_0$ is exact except at $F_i(\Delta) = C_m(\Delta)$, where

$$\tilde{H}_m(\Delta) = \text{Ker} \partial_m / \text{Im} \partial_0 = \text{Im} \partial_m$$

which is

$$\text{span}_\mathbb{R} \{ e_1, e_2, \ldots, e_m \}$$

So $\tilde{H}_i(S^{n-1}) = \begin{cases} \mathbb{R} & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}$
Two topological spaces \( X \) and \( Y \) are **homeomorphic** if there is \( f: X \rightarrow Y \) such that:
- \( f \) is bijective
- \( f \) is continuous
- \( f^{-1} \) is continuous

Think: deform \( X \) to \( Y \) continuously.

**Ex:**
- \( \mathbb{R}^2, \mathbb{C} \): yes
- \( \mathbb{R}, \mathbb{R}^2 \): yes
- \( \mathbb{R}^2, \mathbb{R}^2 \): no (\( m \neq n \))
- \( \mathbb{R} \times \mathbb{R} \): n-simplex: no (\( n \geq 1 \))

Then if \( X \) and \( Y \) are homeomorphic, they have the same homology groups.

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Then if \( X \) and \( Y \) are homotopy equivalent then they have the same homology groups.

**Ex:**

\[ X = B^n = \left\{ x \in \mathbb{R}^n \mid \left\| x \right\|_2 \leq 1 \right\} \]

\[ Y = \{0\} \subset \mathbb{R}^n \]

Let \( f: X \rightarrow Y \)

\[ g: Y \rightarrow X \]

**Then**

\[ f \circ g: Y \rightarrow X \]

\[ g \circ f: X \rightarrow X \]

\[ x \mapsto 0 \quad 0 \mapsto 0 \]

\[ x \mapsto 0 \quad x \mapsto 0 \]

\[ \text{is the identity} \quad \text{is homotopic to the identity} \]

Since \( B^n \) and \( \Delta^n \) are homotopic

\( B^n \) and \( \Delta^n \) are homotopic to the identity

Then \( \tilde{H}_i(\Delta^n) = \tilde{H}_i(*) = 0 \) for all \( i \).

So the chain complex for \( \Delta^{n+1} \) is exact:

\[ 0 \rightarrow \mathbb{F} \overset{\partial_n}{\longrightarrow} \mathbb{F} \overset{\partial_{n-1}}{\longrightarrow} \cdots \overset{\partial_2}{\longrightarrow} \mathbb{F} \overset{\partial_1}{\longrightarrow} \mathbb{F} \overset{\partial_0}{\longrightarrow} 0 \]

Hence the chain complex for \( \partial \Delta^{n-1} = \Sigma^{n-2} \):

\[ 0 \rightarrow \mathbb{F} \overset{\partial^{n-1}}{\longrightarrow} \cdots \overset{\partial_2}{\longrightarrow} \mathbb{F} \overset{\partial_1}{\longrightarrow} \mathbb{F} \overset{\partial_0}{\longrightarrow} 0 \]

It is exact except at \( C_1 \) when \( \tilde{H}_1(\partial \Delta^n) = \mathbb{F} \).

This gives another proof for the homology of the sphere.