Ok, so let us focus (for a while!) on **MONOMIAL IDEALS**

(following Miller-Sturmfels)

A **monomial ideal** \( I \subseteq \mathbb{F}[x] = \mathbb{F}[x_1, \ldots, x_n] \) is one generated by monomials:

\[ I = \langle x^{a_1}, \ldots, x^{a_n} \rangle \]

There is a unique minimal generating set. It is finite.

**Square-Free Monomial Ideals** (Stanley-Reisner)

- \( x^a \) is square-free if each \( a_i \in \{0,1\} \)
- \( I \) is square-free if its generators are.

Ex. \( I = \langle ad, ae, bcd, be, ce, de \rangle \) in \( \mathbb{F}[a,b,c,d,e] \)

let \( \Delta_3 \) = \{ supports of square-free monomials not in \( I \) \}

Here \( \Delta_3 = \{1,0,4,4,d,e,ab,ac,be,bd,cd,de,abc\} \) are the faces of the simplicial complex.

Def An abstract simplicial *complex* on \( E \) is a collection \( \mathcal{F} \) of subsets of \( E \) "faces/simplices" with (GE \( \subseteq \) \( \mathcal{F} \)) \( \Rightarrow \) \( \mathcal{F} \subseteq \) say dim \( E = 1F1-1 \).
Think: points, lines, triangles, tetrahedra, ... simplices

glued along their common faces

**Theorem**

There is a bijection

\[(\text{simplicial complex}) \leftrightarrow (\text{saturated monomial})\]

on \([n]\)

\[\text{ideals in } \mathbb{K}[x_1, \ldots, x_n]\]

**Pf.**

Given \(\Delta\) a simplicial complex, let

\[I_\Delta = \langle x_{i_1} \cdots x_{i_k} \mid \{i_1, \ldots, k\} \in \Delta \rangle\]

Given an ideal \(I\), let

\[\Delta(I) = \{\{i_1, \ldots, k\} \mid x_{i_1} \cdots x_{i_k} \in I\}\]

This is clearly made of each other.

**Def**

\[I_\Delta = \text{Stanley-Reisner ideal of } \Delta\]

\[R/I_\Delta = \text{Stanley-Reisner ring of } \Delta\]

Let \(m_\sigma = \langle x_i \mid i \in \sigma \rangle\) for \(\sigma \subset [n]\) (prime ideal)

**Prop**

\[I_\Delta = \bigcap_{\sigma \in \Delta} m_{\sigma-\sigma}\]

**Pf.** \(\subseteq\):

Let \(x_\sigma = \prod_{i \in \sigma} x_i\) be a gen of \(LHS\), \(\tau \in \Delta\).

\[\sup x_\sigma \notin M^{\tau - \tau} = \langle x_{\alpha_1}, \ldots, x_{\alpha_k} \rangle \supseteq \Delta\]

Then \(\alpha_1 \notin \tau, \ldots, \alpha_k \notin \tau\), so \(\tau - \sigma \subset \tau - \tau\)

so \(\tau \subseteq \sigma \Rightarrow \tau \subseteq \sigma\)

\(\supseteq\):

**RHS is monomial.

A monomial \(x_\alpha^a \cdots x_\alpha^a\) is in RHS iff

iff \(\{\alpha_1, \ldots, \alpha_k\}\) contains at least one el of all \(\tau - \tau\)

iff \(\{\alpha_1, \ldots, \alpha_k\} \notin \tau\) for all \(\sigma \in \Delta\).**

**Ex.**

\[
\begin{align*}
I_\Delta &= \langle ac, be, bcd, le, ce, de \rangle & \text{(min monom)} \\
&= \langle d, e \rangle \cap \langle a, b, e \rangle \cap \langle a, c, e \rangle \cap \langle a, b, d \rangle \\
&= \langle c, e \rangle \cap \langle c, e \rangle \cap \langle c, d \rangle \cap \langle a, d \rangle & \text{(max monom)}
\end{align*}
\]

**Note**

If \(\mathbb{K}\) is infinite, the following are in bijection:

- simplicial complex on \([n]\)
- saturated monomial ideals in \(\mathbb{K}[x_1, \ldots, x_n]\)
- union of coordinate subspaces in \(\mathbb{K}^n\)

(In ex, subspaces are \(abc, cd, bd, e\).)