Corollary (M. Vaserstein) A graded $R$-module $M$ has

$$H(M; x) = \frac{K(M; x)}{(1-x)^n}$$

$H_M(d)$ equals a polynomial in $d$ of degree $\leq n$
for large enough $d$.

In ex: $1 = \frac{1}{(1-x)^3} - 3 \frac{x}{(1-x)^2} + 3 \frac{x^2}{1-x} - \frac{x^3}{(1-x)^2}$

In practice, this works better.

Theorem (Macaulay)

Let $P$ be a fin. gen. graded $R$-module, given as $P = F/M$, $F$ free module with homog. basis $M = $ submodule gen. by homog. elt's (
relatios).

Then $F/M$ and $F/M$ have the same Hilbert function.

Proof: Let $B = \{ \text{monomials in } F \text{, not in } m(M) \}$

Claim: $B$ is a basis for $P = F/M$
$B_{dt}$ is a basis for $P_{dt}$

Proof: Suppose $\sum_i \lambda_i m_i = 0$ in $P$ $m_i \in B$
$\sum_i \lambda_i m_i - \sum_i m_i = m \in M$

so $m(M)$ is one of the $m_i$
$\Rightarrow m_1 \in B$.

Span. Suppose $f \in F/M$ is not gen. by $B$

Then $f \in F$ is not gen. by $M_{\mu B}$.

Take such an $f$ with $\deg(f)$ minimal.

If $\deg(f) \in B$, then
$g = f - \deg(f)$ is not gen. by $M_{\mu B}$
has $\deg(g) < \deg(f)$.

If $\deg(f) \notin B$ then $\deg(f) \notin M$
so take $m \in M, m = \deg(f) + m'$

$f = \deg(f) + f'$

$\deg(f - m' + m'')$

not gen. Smaller by $M_{\mu B}$ initial term.

Note. $F/M \cong \oplus R/I_i$, so

Computing Hilb (arbitrary module) is reduced to
Computing Hilb ($R/I$) for monomial ideals.

This is NP-hard (Bayer-Stallman), but can be done reasonably for small/"nice" $I$. 


A useful procedure for $H_{R/I}(n)$:

Let $I = \langle m_1, \ldots, m_k \rangle$.

Define: $n = m_k$, $d = \text{deg } n$

$I' = \langle m_1, \ldots, m_{k-1} \rangle$

$J = (I': m_k) = \{ m \mid m \in I' \}$

$= \langle \frac{m_1}{\gcd(m_1,n)}, \ldots, \frac{m_{k-1}}{\gcd(m_{k-1},n)} \rangle$

Then there is an exact sequence of $R$-modules:

$0 \to (R/J)(-d) \to R/I' \to R/I \to 0$

so

$H_{R/I}(n) = H_{R/I'}(n) - H_{R/J}(n-d)$

(I' has fewer generators.)

(HW Exercise.)

There are other procedures.

(By the way:

If $I$ is a homogeneous ideal in $R$,

$H(R/I; x) = H(R; x) - H(I; x)$

$H(R/I; x) = \frac{1}{(1-x)^n} - H(I; x)$

So computing the Hilbert function of $I, R/I$ are equivalent questions.)

This allowed Macaulay to completely classify the Hilbert functions of graded rings (with some conditions).

Fact

For fixed $n,k$, there is a unique expression

$n = \binom{ak}{k} + \binom{ak-1}{k-1} + \cdots + \binom{ai}{i} \quad a_k > a_{k-1} > \cdots > a_i \geq 1$

Let

$\gamma^k(n) = \binom{ak-1}{k-1} + \binom{ak-2}{k-2} + \cdots + \binom{ai-1}{i-1}$

Theorem (Macaulay)

For $(f_0, f_1, \ldots) \in N^{\infty}$ the following are equivalent:

(i) $f_0 = 1$ and $\gamma^k(f_k) \leq f_{k-1}$ for $k \geq 2$

(ii) There exists a graded ring $R$ with $R_0 = \mathbb{F}$ (field) and $R_1$ generating $R$, so that

$H_2(i) = f_i$

(iii) There is a multicomplex with $f_k = f_k$.

(A multicomplex $M$ is a set of monomials $\{m_{ni} \mid m_{ni} \in M \}$ such that $m_{ni} \in M$, $n_{nl} \in M$. Its frusto is $(f, h_{-1})$ where $f_k = \#$ of monomials of deg $i$.)

Ex: $(1, 3, 6, 4, 1)$ (Many more variants of this result.)