Subdominant matroid ultrametrics

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Abstract

Given a matroid M on the ground set E, the Bergman fan $\widetilde{\mathcal{B}}(M)$, or space of M-ultrametrics, is a polyhedral complex in $\mathbb{R}^E$ which arises in several different areas, such as tropical algebraic geometry, dynamical systems, and phylogenetics. Motivated by the phylogenetic situation, we study the following problem: Given a point ω in $\mathbb{R}^E$, we wish to find an M-ultrametric which is closest to it in the ℓ_∞-metric.

The solution to this problem follows easily from the existence of the subdominant M-ultrametric: a componentwise maximum M-ultrametric which is componentwise smaller than ω. A procedure for computing it is given, which brings together the points of view of matroid theory and tropical geometry.

When the matroid in question is the graphical matroid of the complete graph K_n, the Bergman fan $\widetilde{\mathcal{B}}(K_n)$ parameterizes the equidistant phylogenetic trees with n leaves. In this case, our results provide a conceptual explanation for Chepoi and Fichet's method for computing the tree that most closely matches measured data.

1 Introduction

Given a matroid M on the ground set E, the Bergman fan $\widetilde{\mathcal{B}}(M)$, or space of M-ultrametrics, is a polyhedral complex in $\mathbb{R}^E$ which arises in several different areas, such as tropical algebraic geometry [12], dynamical systems [8] and phylogenetics [1]. It has been described topologically and combinatorially [1]. Motivated by the phylogenetic situation, we study the following problem: Given a point ω in $\mathbb{R}^E$, we wish to find an M-ultrametric which is closest to it in the ℓ_∞-metric.

In Section 2 we define the Bergman fan $\widetilde{\mathcal{B}}(M)$, as well as the notion of an M-ultrametric. We offer several characterizations of them. When $M(K_n)$ is the graphical matroid of the complete graph K_n, $M(K_n)$-ultrametrics are precisely ultrametrics in the usual sense.

In Section 3 we show that, in fact, there is a componentwise maximum M-ultrametric which is componentwise smaller than ω. We call it the *subdominant M-ultrametric* of ω, and denote it ω^M. A simple translation of it is a closest M-ultrametric to ω. A procedure for computing subdominant M-ultrametrics is given, similar in spirit to Tarjan and Kozen's blue and red rules [9, 14] for computing the bases of minimum weight of a matroid.

In Section 4 we prove that the Bergman fan is a tropical polytope in the sense of [7], and that the subdominant M-ultrametric ω^M is precisely the tropical projection of ω onto $\widetilde{\mathcal{B}}(M)$.

In Section 5, we discuss a special case of particular importance: the Bergman fan of $M(K_n)$, the graphical matroid of the complete graph K_n. As shown by Ardila and Klivans [1], the Bergman fan $\widetilde{\mathcal{B}}(K_n)$ can be regarded as a space of phylogenetic trees, and we are interested in finding an (equidistant) phylogenetic tree that most closely matches measured data. In this case, our results provide a conceptual explanation for the method developed by Chepoi and Fichet in [6] to compute the tree that most closely matches measured data.

Throughout this paper, familiarity with the fundamental notions of matroid theory will be very useful; we refer the reader to [10, Ch. 1,2] for the basic definitions.

2 The Bergman fan and matroid ultrametrics

Let M be a matroid of rank r on the ground set E, and let $\omega \in \mathbb{R}^E$. Regard ω as a weight function on M, so that the weight of a basis $B = \{b_1, \ldots, b_r\}$ of M is given by $\omega_B = \omega_{b_1} + \omega_{b_2} + \cdots + \omega_{b_r}$.

Let M_ω be the collection of bases of M having minimum ω-weight. This collection is itself the set of bases of a matroid; for more information, see for example [1].

Definition. *The* Bergman fan *of a matroid M with ground set E is:*

$$\widetilde{\mathcal{B}}(M) := \{\, \omega \in \mathbb{R}^E \ : \ \textit{every element of } E \textit{ is in a } \omega\textit{-minimum basis}\}.$$

*An M-*ultrametric *is a vector in $\widetilde{\mathcal{B}}(M)$.*

Our ongoing example throughout the paper will be the Bergman fan of the matroid $M(K_4)$, the graphical matroid of the complete graph K_4. An $M(K_4)$-ultrametric is an assignment of weights to the edges of K_4 such that any edge of K_4 is in a spanning tree of minimum weight. An example of an $M(K_4)$-ultrametric is given in Figure 1. The ω-minimum spanning trees are

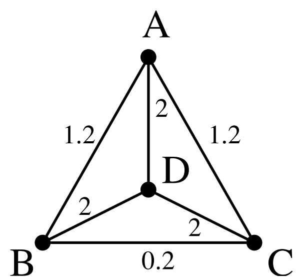

Figure 1: An $M(K_4)$-ultrametric.

those consisting of an edge of weight 2, an edge of weight 1.2, and an edge of weight 0.2. Every edge of the graph is in at least one such tree.

We start by reviewing some useful facts about matroids. Given a basis B and an element $x \notin B$, there is a unique circuit which is contained in $B \cup x$ (and must contain x). It is called the *fundamental circuit of B and x.*

Given a basis B and an element $y \in B$, there is a unique cocircuit which is disjoint with $B - y$ (and must contain y). It is called the *fundamental cocircuit of B and y.* It is equal to the fundamental circuit of $E - B$ and y in the dual matroid M^*.

Proposition 1. *[5, Lemma 7.3.1] Let B be a basis of M, and let $x \notin B$ and $y \in B$. The following are equivalent.*

(i) $B \cup x - y$ is a basis.

(ii) y is in the fundamental circuit of B and x.

(iii) x is in the fundamental cocircuit of B and y.

We will also need the following lemma:

Lemma. *[10, Prop. 2.1.11] A circuit and a cocircuit cannot intersect in exactly one element.*

We now give two additional characterizations of the Bergman fan, which will be central to our analysis.

Proposition 2. *Let M be a matroid with ground set E, and let $\omega \in \mathbb{R}^E$. The following are equivalent:*

(i) ω is an M-ultrametric.

(ii) No circuit has a unique ω-maximum element.

(iii) Every element of E is ω-minimum in some cocircuit.

Proof. $(ii) \Rightarrow (i)$: Let $e \in E$; we want to show that it is in an ω-minimal basis. Consider any ω-minimal basis B. If $e \in B$, there is nothing to prove. Otherwise, the fundamental circuit of B and e has at least two w-maximum elements; let $f \neq e$ be one of them. By Proposition 1, $B \cup e - f$ is a basis; since $\omega_f \geq \omega_e$, it is actually the ω-minimum basis that we need.

$(iii) \Rightarrow (ii)$: Assume that a circuit C achieves its ω-maximum only at e. Say C^* is a cocircuit where e is ω-minimum. By the Lemma, we can find an element $f \neq e$ in $C \cap C^*$; then $\omega_e > \omega_f$ because e is the unique ω-maximum in C, and $\omega_e \leq \omega_f$ because e is ω-minimum in C^*.

$(i) \Rightarrow (iii)$: Let $e \in E$; we want to show that it is ω-minimum in some cocircuit. Let B be an ω-minimum basis containing e, and let f be the ω-minimum element in the fundamental cocircuit C^* of B and e. Then $B \cup f - e$ is a basis, and its weight is at least ω_B. Therefore $\omega_f \geq \omega_e$, so e is ω-minimum in C^* also.

$\square$

Let us check Proposition 2 for the $M(K_4)$-ultrametric of Figure 1. Statement (ii) is clear: in each cycle of K_4, the two largest weights are equal. To check statement (iii), recall that the cocircuits of $M(K_4)$ are the cuts of K_4. Denote by $S - S'$ the cut that separates the vertices in S from the vertices in S'. Then the edges of weight 2 are minimum in the cut $ABC - D$, the edges of weight 1.2 are minimum in the cut $A - BCD$, and the edge of weight 0.2 is minimum in the cut $AB - CD$.

3 Subdominant M-ultrametrics.

For $\omega, \omega' \in \mathbb{R}^E$, say that $\omega \leq \omega'$ if $\omega_e \leq \omega'_e$ for each $e \in E$.

Proposition. *Let M be a matroid on E, and let $\omega \in \mathbb{R}^E$. There exists a unique maximum M-ultrametric which is less than or equal to ω. We call it the* subdominant M-ultrametric *of ω, and denote it ω^M.*

Proof. Let $S = \{\omega' \in \widetilde{\mathcal{B}}(M) \ : \ \omega' \leq \omega\}$, and let $\omega^M = \sup S$, where the sup is taken componentwise. We claim that $\omega^M \in \widetilde{\mathcal{B}}(M)$.

Proceed by contradiction; assume that e is the unique ω^M-maximum of the circuit C. Let ϵ be such that $\omega^M_e - \epsilon > \omega^M_f$ for all $f \in C - e$. We can find a $\omega' \in S$ such that $\omega'_e > \omega^M_e - \epsilon$. But then $\omega'_e > \omega^M_f \geq \omega'_f$ for all $f \in C - e$, so e is the unique ω'-maximum of C. This contradicts the assumption that $\omega' \in \widetilde{\mathcal{B}}(M)$. $\square$

We now turn to the problem of constructing the subdominant M-ultrametric of a vector ω. Take a vector ω which is not an M-ultrametric. This means that conditions *(ii)* and *(iii)* of Proposition 2 are not satisfied.

We must force each element to be ω-minimum in some cocircuit, and not allow it to be the unique ω-maximum element of a circuit. At the same time, we must demand that it keeps its weight as high as possible. Let us issue the following two rules, similar in spirit to Tarjan's *blue and red rules* for constructing the minimum weight spanning trees of a graph. [9, 14]

Blue Rule: Suppose an element e is not ω-minimum in any cocircuit. Look at the minimum weight in each cocircuit containing e. (They are all less than ω_e.) Make ω_e equal to the largest such weight.

Red Rule: Suppose an element e is the unique ω-maximum element of a circuit. Look at the maximum weight in $C - e$ for each circuit C containing e. (At least one of them is less than ω_e.) Make ω_e equal to the smallest such weight.

Figure 2 shows an example of an assignment ω of weights to the edges of K_4 which is not an ultrametric, and the result of applying the blue rule or the red rule to edge CD. When the blue rule is applied, the edge CD inherits its new weight from the cut $ABC - D$, as shown. When the red rule is applied, the edge CD inherits its new weight from either one of the cycles ACD (as shown) or $ABCD$. Notice that, surprisingly, the blue rule and the red rule give the same result. The reason for this will soon be explained.

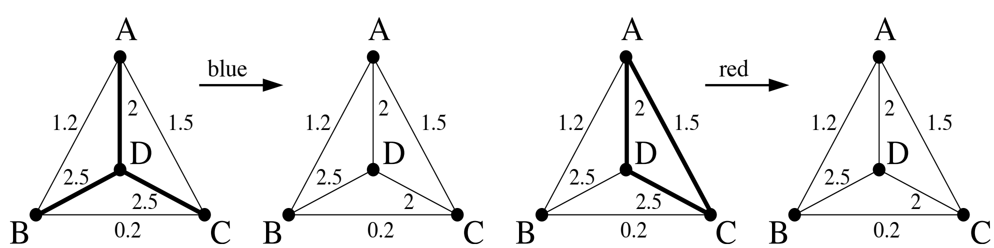

Figure 2: Applying the blue rule or the red rule to edge CD.

In principle, it is not yet clear how to apply these rules, or what they will do. We do not know in what order we should apply them, and different orders would seem likely to give different results. It is not even obvious that applying these two local rules successively will accomplish the global goal of turning ω into an M-ultrametric.

Fortunately, as the following theorem shows, the situation is much simpler than expected. It turns out that, if each element of E is willing to do its part by complying with the two rules, the global goal of constructing the subdominant M-ultrametric will inevitably be achieved.

Theorem 1. *Let M be a matroid on the ground set E, let $\omega \in \mathbb{R}^E$. For each element $e \in E$, there are two possibilities:*

1. *Both the blue rule and the red rule apply to e, and they both change its weight from ω_e to ω_e^M, or*
2. *Neither the blue rule nor the red rule apply to e, and $\omega_e = \omega_e^M$.*

Consequently, if we apply the blue rule or the red rule to each element of E in any order, we obtain the subdominant M-ultrametric ω^M.

Before proving Theorem 1, let us illustrate it with an example: the construction of the subdominant ultrametric of the weight vector ω we considered in Figure 2. Figure 3 shows the result of applying, at each step, either the blue rule or the red rule to the highlighted edge. The reader is invited to check that, at each step of the process, the blue rule and the red rule give the same result.

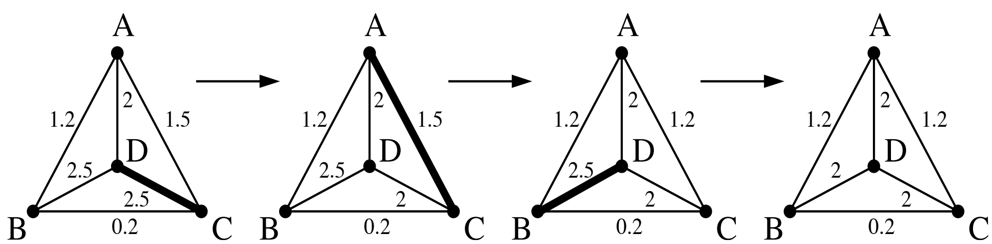

Figure 3: Constructing the subdominant $M(K_4)$-ultrametric of ω.

After three steps, we reach an ultrametric. This means that we are done: we have in fact reached $\omega^{M(K_4)}$, the subdominant $M(K_4)$-ultrametric of ω. Moreover, we could have applied *either the blue rule or the red rule* to the edges of K_4 *in any order*, and we would have obtained exactly the same result: $\omega^{M(K_4)}$.

Proof of Theorem 1. Let ω'_e and ω''_e be the weights assigned to e by the blue rule and the red rule, respectively. If the blue rule (or the red rule) does not apply to e, set $\omega'_e = \omega_e$ (or $\omega''_e = \omega_e$). We proceed in several steps.

1. $\omega \geq \omega' \geq \omega^M$ and $\omega \geq \omega'' \geq \omega^M$.

The inequalities $\omega_e \geq \omega'_e \geq \omega_e^M$ and $\omega_e \geq \omega''_e \geq \omega_e^M$ are easy to see: When one of the rules applies, it decrees a decrease in the weight of e that is clearly necessary for any M-ultrametric which is less than ω.

2. ω' is an ultrametric.

We use Proposition 2(iii). Let $e \in E$. By definition, ω'_e is equal to the ω-minimum weight of some cocircuit C^* containing e. (This is true even if the blue rule does not apply.) We claim that it is also the ω'-minimum weight of C^*. Let $f \in C^*$. Then ω'_f is the largest ω-minimum weight of a cocircuit containing f; this is greater than or equal to the ω-minimum weight of C^*, which is ω'_e.

3. $\omega' = \omega^M$.

We already knew that $\omega' \geq \omega^M$. Now, since ω' is an ultrametric less than or equal to ω, it must be less than or equal to ω^M by Proposition 3.

4. $\omega'' = \omega^M$.

We already knew that $\omega'' \geq \omega^M$. Now assume that $\omega''_e > t > \omega'_e$. Let $S = \{f \in E \,|\, \omega_f < t\}$. We have $\omega_e \geq \omega''_e > t$, so e is not in S. Since $\omega''_e > t$, no circuit containing e is in $S \cup e$. Therefore $e \notin \mathrm{cl}(S)$, and there is a hyperplane H containing $\mathrm{cl}(S)$ with $e \notin H$. The complement of H is a cocircuit C^* which contains e, and does not contain any element of S; its ω-minimum weight is at least t. Therefore $\omega'_e \geq t$, a contradiction. $\square$

If we have an ω-minimum basis of M, which is easy to construct using Tarjan and Kozen's blue and red rules [14, 9], the task of constructing ω^M becomes even simpler, as the following result shows.

Proposition. *Let B be any ω-minimum basis.*

(i) For each $e \in B$, $\omega_e^M = \omega_e$.

(ii) For each $e \notin B$, let C_0 be the fundamental circuit of B and e. Then ω_e^M is equal to the ω-maximum weight in $C_0 - e$.

Proof. It follows from Proposition 1 that any element $e \in B$ is ω-minimum in the fundamental cocircuit of B and e. Therefore the blue rule does not apply to e, and *(i)* follows from Theorem 1.

Now we show that, for $e \notin B$, $\min_{e \in C} \max_{j \in C-e} \omega_j = \max_{j \in C_0 - e} \omega_j$. Say the maximum in the right hand side is achieved at ω_m. First we claim that, for any circuit C containing e, there is a basis $B - m \cup f$ with $f \in C - e$. In fact, $B - m \cup C$ has full rank, since it contains the basis $B - m \cup e$. But

e is dependent on $C-e$, and hence on $(B-m)\cup(C-e)$ also. Therefore $(B-m)\cup(C-e)$ has full rank as well, and the claim follows. Now, since $B-m\cup f$ is a basis, $\omega_m \leq \omega_f \leq \max_{j\in C-e}\omega_j$. The desired equality follows.

Now *(ii)* follows easily. If the red rule applies to e, it will change ω_e to $\omega_e^M = \omega_m$. If it does not apply, then e is not the unique ω-max of any circuit, and $\omega_e^M = \omega_e = \omega_m$. $\square$

4 Tropical projection

In this section we study the Bergman complex of a matroid from the point of view of tropical geometry. We start by reviewing the basic definitions; for more information, we refer the reader to [7].

The *tropical semiring* $(\mathbb{R}\cup\{\infty\},\oplus,\odot)$ is the set of real numbers augmented by infinity, together with the operations of *tropical addition and multiplication*, which are defined by $x\oplus y = \min(x,y)$ and $x\odot y = x+y$.

Throughout this section, the symbol $\mathbb{R}$ will denote the tropical semiring, and not the ring of real numbers. The additive and multiplicative units of $\mathbb{R}$ are ∞ and 0, respectively.

The n-dimensional space $\mathbb{R}^n$ is a semimodule over the tropical semiring, with addition $x\oplus y$ and scalar multiplication $c\odot x$ given componentwise. Tropical scalar multiplication by c is equivalent to translation by the vector $(c,c,\ldots,c)$. The $(n-1)$*-dimensional tropical projective space* is $\mathbb{TP}^{n-1} = \mathbb{R}^n/(x\sim(c\odot x))$.

A subset S of $\mathbb{R}^n$ is *tropically convex* if $(a\odot x)\oplus(b\odot y)\in S$ for any $x,y\in S$ and *any* $a,b\in\mathbb{R}$. Tropically convex sets are invariant under tropical scalar multiplication; *i.e.*, translation by $(c,c,\ldots,c)$. We will therefore identify them with their image in $\mathbb{TP}^{n-1}$. A *tropical polytope* is the set

$$\operatorname{tconv}(V) = \{(a_1\odot v_1)\oplus\cdots\oplus(a_r\odot v_r)\ :\ a_1,\ldots,a_r\in\mathbb{R}\}$$

of all linear combinations of a finite set $V=\{v_1,\ldots,v_r\}\subseteq\mathbb{TP}^{n-1}$.

Our claim is that, for any matroid M on the set $[n]$, $-\widetilde{\mathcal{B}}(M)$ is a tropical polytope in $\mathbb{TP}^{n-1}$. For each flat F of M, let $v_F\in\mathbb{TP}^{n-1}$ be the vector whose i-th coordinate is ∞ if $i\in F$, and 0 otherwise. Recall that the *hyperplanes* of a matroid are its maximal proper flats.

Proposition. *For any matroid M on $[n]$, $-\widetilde{\mathcal{B}}(M)$ is a tropical polytope in $\mathbb{TP}^{n-1}$. Its set of vertices is*

$$V_M = \{v_H\ :\ H \textit{ is a hyperplane of } M\}.$$

Proof. We start by reviewing the description of the Bergman fan obtained in [1]. Given a flag of subsets $\mathcal{F} = \{\emptyset =: F_0 \subset F_1 \subset \cdots \subset F_k \subset F_{k+1} := E\}$, the *weight class* of $\mathcal{F}$ is the set of $\omega \in \mathbb{R}^n$ for which ω is constant on each set $F_i - F_{i-1}$, and has $\omega|_{F_i - F_{i-1}} < \omega|_{F_{i+1}-F_i}$. For example, one of the weight classes in $\mathbb{R}^5$ is the set of vectors ω such that $\omega_1 = \omega_4 < \omega_2 < \omega_3 = \omega_5$. It corresponds to the flag $\{\emptyset \subset \{1,4\} \subset \{1,2,4\} \subset \{1,2,3,4,5\}\}$.

The disjoint union of all weight classes is $\mathbb{R}^n$. The Bergman fan $\widetilde{\mathcal{B}}(M)$ is also a union of disjoint weight classes: the weight class of $\mathcal{F}$ is in $\widetilde{\mathcal{B}}(M)$ if and only if $\mathcal{F}$ is a flag of flats of M.

Now we will show that

$$-\widetilde{\mathcal{B}}(M) = \text{tconv}\{v_F \ : \ F \text{ is a proper flat of } M\}.$$

The left hand side is contained in the right hand side because if $\mathcal{F}$ is a flag of flats, the negative of a vector w in the weight class of $\mathcal{F}$ can be obtained as

$$-\omega = (-\omega|_{F_1} \odot v_{F_0}) \oplus (-\omega|_{F_2 - F_1} \odot v_{F_1}) \oplus \cdots \oplus (-\omega|_{F_r - F_{r-1}} \odot v_{F_{r-1}}).$$

To see that the right hand side is contained in $-\widetilde{\mathcal{B}}(M)$, since $v_F \in -\widetilde{\mathcal{B}}(M)$ for any flat F, it suffices to show that $-\widetilde{\mathcal{B}}(M)$ is tropically convex.

A consequence of the previous description of $\widetilde{\mathcal{B}}(M)$ is the following. A vector $x \in \mathbb{R}^n$ is in $-\widetilde{\mathcal{B}}(M)$ if and only if, for any $r \in \mathbb{R}$, the set $(x; r) = \{i \in [n] : x_i \geq r\}$ is a flat of M. Now take $x, y \in -\widetilde{\mathcal{B}}(M)$ and $a, b \in \mathbb{R}$, and let $z = (a \odot x) \oplus (b \odot y)$, so $z_i = \min(a + x_i, b + y_i)$. Then, for any $r \in \mathbb{R}$, we have that $(z; r) = (x; r - a) \cap (y; r - b)$. This is a flat in M, because both $(x; r - a)$ and $(y; r - b)$ are flats. Thus $z \in -\widetilde{\mathcal{B}}(M)$.

Finally, we prove the claim about the vertices of $-\widetilde{\mathcal{B}}(M)$. If $F = F_1 \cap \cdots \cap F_k$ is an intersection of larger flats, then

$$v_F = (0 \odot v_{F_1}) \oplus \cdots \oplus (0 \odot v_{F_k})$$

so v_F is not a vertex. Conversely, suppose that we have an equation

$$v_F = (a_1 \odot v_{F_1}) \oplus \cdots \oplus (a_k \odot v_{F_k}).$$

We can assume that $a_i \neq \infty$ for all i. For each $f \in F$, $(v_F)_f = \infty$. Thus for all i we have $(v_{F_i})_f = \infty$; *i.e.*, $f \in F_i$. For each $\bar{f} \notin F$, $(v_F)_{\bar{f}} = 0$. Thus for some i we have $(v_{F_i})_{\bar{f}} \neq \infty$; *i.e.*, $\bar{f} \notin F_i$. We conclude that $F = F_1 \cap \cdots \cap F_k$. $\square$

For any tropical polytope P in $\mathbb{TP}^{n-1}$, Develin and Sturmfels [7] gave an explicit construction of a *nearest point map* $\pi_P : \mathbb{TP}^{n-1} \to P$, which maps every point in tropical projective space to a point in P which is closest to it in the ℓ_∞-metric:

$$||\, x - y||_\infty = \max_{1 \le i,j \le n} |(x_i - x_j) - (y_i - y_j)|.$$

This nearest point map is given as follows. Let $x \in \mathbb{TP}^{n-1}$. For each vertex v of P, we need to compute λ_v: the minimum λ such that $(\lambda \odot v) \oplus x = x$. Then

$$\pi_P(x) = \bigoplus_{v \text{ vertex of } P} (\lambda_v \odot v)\,.$$

Proposition 3. *The tropical projection $\pi_{-\widetilde{\mathcal{B}}(M)}$ maps each vector ω to its subdominant M-ultrametric:*

$$\pi_{-\widetilde{\mathcal{B}}(M)}(-\omega) = -\omega^M.$$

Proof. In our case, it is easy to see that $\lambda_{v_H} = \max_{f \notin H} \omega_f$. Therefore

$$\begin{aligned} \pi(-\omega)_e &= \min_{\{H\,:\, e \notin H\}} \max_{f \notin H} -\omega_j \\ &= -\max_{e \in C^*} \min_{f \in C^*} \omega_f, \end{aligned}$$

remembering that the cocircuits of M are precisely the complements of its hyperplanes. This last expression is the result of applying the blue rule to element e. ☐

5 Phylogenetic trees

Theorem 1 is of particular interest when applied to $M(K_n)$, the graphical matroid of the complete graph K_n. As shown in [1], the Bergman fan $\widetilde{\mathcal{B}}(K_n)$ can be regarded as a space of phylogenetic trees. Our results thus provide a new point of view on a known algorithm in phylogenetics. In fact, it is this context that provided the original motivation for our results.

Let us now review the connection between the Bergman fan $\widetilde{\mathcal{B}}(K_n)$, ultrametrics, and phylogenetic trees. For more information, see [1].

Definition. *A* dissimilarity map *is a map $\delta : [n] \times [n] \to \mathbb{R}$ such that $\delta(i,i) = 0$ for all $i \in [n]$, and $\delta(i,j) = \delta(j,i)$ for all $i,j \in [n]$. An* ultrametric *is a dissimilarity map such that, for all $i,j,k \in [n]$, two of the values $\delta(i,j), \delta(j,k)$ and $\delta(i,k)$ are equal and not less than the third.*

We can think of a dissimilarity map $\delta : [n] \times [n] \to \mathbb{R}$ as a weight function $\omega_\delta \in \mathbb{R}^{\binom{[n]}{2}}$ on the edges of the complete graph K_n. The connection with our study is given by the following result.

Theorem. *[1] A dissimilarity map δ is an ultrametric if and only if ω_δ is a $M(K_n)$-ultrametric.*

As mentioned earlier, the previous theorem is our reason for giving vectors in $\widetilde{\mathcal{B}}(M)$ the name of M-ultrametrics. We will slightly abuse notation and write δ instead of ω_δ; this should cause no confusion.

Ultrametrics are also in correspondence with a certain kind of phylogenetic tree. Let T be a rooted metric n-tree; that is, a tree with n leaves labelled $1, 2, \ldots, n$, together with a length assigned to each one of its edges. For each pair of leaves u, v of the tree, we define the *distance* $d_T(u, v)$ to be the length of the unique path joining leaves u and v in T. This gives us a distance function $d_T : [n] \times [n] \to \mathbb{R}$. We are interested in *equidistant n-trees.* These are the rooted metric n-trees such that the leaves are equidistant from the root, and the lengths of the interior edges are positive. (For technical reasons, the edges incident to a leaf are allowed to have negative lengths.)

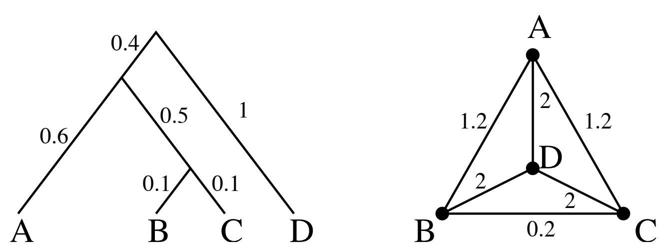

Figure 4: An equidistant tree and its distance function.

Figure 4 shows an example of an equidistant 4-tree, where the distance from each leaf to the root is equal to 2. It also shows the corresponding distance function, recorded on the edges of the graph K_4. This distance function is precisely the $M(K_4)$-ultrametric of Figure 1. This is not a coincidence, as the following theorem shows.

Theorem. *[11, Theorem 7.2.5] A dissimilarity map $\delta : [n] \times [n] \to \mathbb{R}$ is an ultrametric if and only if it is the distance function of an equidistant n-tree.*

We can think of equidistant trees as a model for the evolutionary relationships between a certain set of species. The various species, represented by the leaves, descend from a single root. The descent from the root to a leaf tells us the history of how a particular species branched off from the others, until the present day. For more information on the applications of this and other similar models, see for example [4] and [11].

One important problem in phylogenetics is the following: suppose we have a way of estimating, in the present day, the pairwise distances $d_T(i, j)$ between two species. The goal is to recover the most likely tree T. It is well-known how to recover a tree T from its corresponding ultrametric d_T [11, Theorem 7.2.8]. Of course, in real life, the measured data $\delta(i, j)$ will not be exact. No tree will match the measured distances exactly, and we need to find the tree which approximates them most accurately. When proximity is measured in the ℓ_∞ metric:

$$||\delta - d_T||_\infty = \max |\, \delta(i, j) - d_T(i, j)|,$$

Chepoi and Fichet gave a very nice answer to this question, which we now review.

Given a weight function ω on the edges of K_n, let

$$\omega_U(x, y) = \min_{\text{paths } P \text{ from } x \text{ to } y} \; \max_{\text{edges } e \text{ of } P} \omega(e).$$

It is not difficult to see that ω_U is an ultrametric, known as the *subdominant ultrametric* of ω. Notice that $\omega_U = \omega^{M(K_n)}$: the formula above is the result of applying the red rule to edge xy.

Write $2\epsilon = ||\omega - \omega_U||_\infty = \min |\omega(e) - \omega_U(e)|$, and define a second ultrametric by $\omega_U^{+\epsilon}(e) = \omega_U(e) + \epsilon$ for each edge e of K_n.

Theorem. *[6] Given a dissimilarity map ω on $[n]$, an ℓ_∞-optimal ultrametric for ω is the ultrametric $\omega_U^{+\epsilon}$.*

Our discussion of matroid ultrametrics and tropical projection provides a conceptual explanation of the previous theorem. Chepoi and Fichet's goal is to construct an ℓ_∞-closest point to ω in the Bergman fan $\widetilde{\mathcal{B}}(K_n)$. As mentioned in Section 4, we can, in fact, essentially solve this problem for any tropical polytope via tropical projection.

In tropical projective space, an ℓ_∞-optimal ultrametric is $\pi_{\widetilde{\mathcal{B}}(K_n)}(\omega)$, the tropical projection of ω onto $\widetilde{\mathcal{B}}(K_n)$. By Proposition 3, this is also the subdominant $M(K_n)$-ultrametric, $\omega^{M(K_n)}$. Therefore, in real space, an ℓ_∞-optimal ultrametric will be $\omega^{M(K_n)}$, up to a shift by a constant. It is easy to see that the optimal constant is $\frac{1}{2}||\omega - \omega^{M(K_n)}||_\infty = \epsilon$.

6 Acknowledgments

I would like to thank the organizers and participants of the Seminar on Phylogenetic Trees at the University of California, Berkeley during the Fall of 2003, for a wonderful introduction to this interesting subject. I would also like to thank Bernd Sturmfels, who suggested the connection between tropical geometry and Chepoi and Fichet's result, and Lior Pachter and Carly Klivans for very helpful discussions.

References

[1] F. Ardila and C. Klivans, The Bergman complex of a matroid and phylogenetic trees, arxiv:math.CO/0311370, preprint, 2003.

[2] G. Bergman, The logarithmic limit-set of an algebraic variety, *Transactions of the American Mathematical Society* **157** (1971) 459-469.

[3] R. Bieri and J. Groves, The geometry of the set of characters induced by valuations, *J. Reine Angew. Math.* **347** (1984) 168-195.

[4] L. Billera, S. Holmes, and K. Vogtmann, Geometry of the space of phylogenetic trees, *Advances in Applied Mathematics* **27** (2001) 733-767.

[5] A. Björner, The homology and shellability of matroids and geometric lattices, *Matroid Applications*, Cambridge University Press, Cambridge, 1992.

[6] V. Chepoi and B. Fichet, ℓ_∞-approximation via subdominants, *J. Mathematical Psychology* **44** (2000) 600-616.

[7] M. Develin and B. Sturmfels, Tropical Convexity, arXiv:math.MG/0308254, preprint, 2003.

[8] M. Einsiedler, M. Kapranov, D. Lind, and T. Ward, Non-archimedean amoebas, preprint, in preparation.

[9] D. Kozen, *The design and analysis of algorithms,* Springer-Verlag, 1991.

[10] J. G. Oxley, *Matroid theory*, Oxford University Press, New York, 1992.

[11] C. Semple and M. Steel, *Phylogenetics*, Oxford University Press, 2003.

[12] B. Sturmfels, *Solving Systems of Polynomial Equations*, American Mathematical Society, Providence, 2002.

[13] D. Speyer and B. Sturmfels, The tropical Grassmannian, arXiv:math.AG/0304218, preprint, 2003.

[14] R. E. Tarjan, *Data structures and network algorithms*, Society of Industrial and Applied Mathematics, 1983.