The topology of the external activity complex of a matroid

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Abstract

We prove that the external activity complex $\mathrm{Act}_<(M)$ of a matroid is shellable. In fact, we show that every linear extension of LasVergnas's external/internal order $<_{ext/int}$ on M provides a shelling of $\mathrm{Act}_<(M)$. We also show that every linear extension of LasVergnas's internal order $<_{int}$ on M provides a shelling of the independence complex $IN(M)$. As a corollary, $\mathrm{Act}_<(M)$ and M have the same h-vector. We prove that, after removing its cone points, the external activity complex is contractible if M contains $U_{3,1}$ as a minor, and a sphere otherwise.

1 Introduction

Motivation. The *external activity complex* $\mathrm{Act}_<(M)$ of a matroid is a simplicial complex associated to a matroid M and a linear order $<$ on its ground set. This complex arose in work of the first author with Adam Boocher [2]. They started with a linear subspace L of affine space $\mathbb{A}^n$ with a chosen system of coordinates. There is a natural embedding $\mathbb{A}^n \hookrightarrow (\mathbb{P}^1)^n$ into a product of projective lines, and they considered the closure $\widetilde{L}$ of L in $(\mathbb{P}^1)^n$. They proved that many geometric and algebraic invariants of the variety $\widetilde{L}$ are determined by the matroid of L.

As is common in combinatorial commutative algebra, a key ingredient of [2] was to consider the initial ideals $\mathrm{in}_<\widetilde{L}$ under various term orders. These initial ideals are the Stanley-Reisner ideals of the external activity complexes $\mathrm{Act}_<(M)$ under the different linear orders $<$ of the ground set. This led them to consider and describe the complexes $\mathrm{Act}_<(M)$.

The ideals $\mathrm{in}_<\widetilde{L}$ are shown to be Cohen-Macaulay in [2], and the authors asked the stronger question: Are the external activity complexes $\mathrm{Act}_<(M)$ shellable? The purpose of this note is to answer this question affirmatively.

Our results. The facets of $\mathrm{Act}_<(M)$ are indexed by the bases $\mathcal{B}$ of M, and [2] suggested a possible connection between $\mathrm{Act}_<(M)$ and LasVergnas's *internal order* $<_{int}$ on $\mathcal{B}$. [6] Suprisingly, we find that it is the *external/internal order* $<_{ext/int}$ on $\mathcal{B}$, also defined in [6], which plays a key role. Our main result is the following:

Theorem 1.1. Let $M = (E, \mathcal{B})$ be a matroid, and let $<$ be a linear order on the ground set E. Any linear extension of LasVergnas's external/internal order $<_{ext/int}$ of $\mathcal{B}$ induces a shelling of the external activity complex $\mathrm{Act}_<(M)$.

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As a corollary we obtain that these orders also shell the independence complex $IN(M)$, and in fact we show a stronger statement.

Theorem 1.2. Any linear extension of the internal order $<_{int}$ gives a shelling order of the independence complex $IN(M)$.

These theorems are as strong as possible in the context of LasVergnas's active orders. We also obtain the following enumerative corollary.

Theorem 1.3. The h-vector of $\mathrm{Act}_<(M)$ equals the h-vector of M.

It is easy to see that $\mathrm{Act}_<(M)$ is a cone, and hence trivially contractible. It is more interesting to study the *reduced external activity complex* $\overline{Act}_<(M)$, obtained by removing all the cone points of $\mathrm{Act}_<(M)$. Our main topological result is the following.

Theorem 1.4. Let M be a matroid and $<$ be a linear order on its ground set. The reduced external activity complex $\overline{Act}_<(M)$ is contractible if M contains $U_{3,1}$ as a minor, and a sphere otherwise.

There is an embedding of the independence complex $IN(M)$ in $\overline{Act}_<(M)$, and both complexes have the same h-vector. The independence complex of a coloopless matroid is a wedge of spheres, while the external activity complex is contractible or a sphere. Thus $\overline{Act}_<(M)$ can be seen as a topologically simpler model of a matroid M as a simplicial complex.

The paper is organized as follows. In Section 2 we introduce the necessary definitions and preliminaries. In Section 3 we carry out an example in detail, and show that the hypotheses of Theorems 1.1 and 1.2 are best possible. In Section 4 we prove our main Theorem 1.1 on the shellability of the external activity complex $\mathrm{Act}_<(M)$, and Theorem 1.2, which gives many new shellings of the independence complex $IN(M)$. In Section 5 we show that $\mathrm{Act}_<(M)$ and $IN(M)$ have the same h-vector. Finally, in Section 6, we describe the topology of the reduced external activity complex in Theorem 1.4.

2 Preliminaries

In this section we collect the background information on matroids and shellability that we will need to prove our results.

2.1 Matroids

Basic definitions. A *simplicial complex* $\Delta = (E, \mathcal{I})$ is a pair where E is a finite set and $\mathcal{I}$ is a non empty family of subsets of E, such that if $A \in \mathcal{I}$ and $B \subset A$, then $B \in A$. Elements of $\mathcal{I}$ are called *faces* of the complex. The maximal elements of $\mathcal{I}$ are called *facets*. A complex is said to be *pure* if all facets have the same number of elements.

The following is one of many equivalent ways of defining a matroid:

Definition 2.1. A *matroid* $M = (E, \mathcal{I})$ is a simplicial complex such that the restriction of M to any subset of E is pure.

Since there are several simplicial complexes associated to M, we will denote this one $IN(M) = (E, \mathcal{I})$. It is often called the *independence complex* of M.

The two most important motivating examples of matroids are the following.

- (Linear Algebra) Let E be a set of vectors in a vector space, and let $\mathcal{I}$ consist of the subsets of E which are linearly independent. Then one may verify that $(E, \mathcal{I})$ is a matroid. Such a matroid is called *linear* or *representable*.

- (Graph Theory) Let E be the set of edges of an undirected graph G, and let $\mathcal{I}$ consist of the sets of edges which contain no cycle; that is, the forests of G. Once again one may verify that $(E, \mathcal{I})$ is a matroid. Such a matroid is called *graphical*.

For any matroid $M = (E, \mathcal{I})$, it is customary to call the sets in $\mathcal{I}$ *independent*. The facets of a matroid are called *bases*. The set of all bases is denoted $\mathcal{B}$.

Example 2.2. The simplest example of a matroid is the uniform matroid $U_{n,k}$, whose ground set is $[n]$ and whose independent sets are all the subsets of $[n]$ of cardinality at most k. The uniform matroid $U_{3,1}$ is going to play an important role later.

The minimal non-faces of M, that is, the minimal dependent sets, are called *circuits*. The circuits of a matroid have a special structure [7, Lemma 1.1.3]:

Lemma 2.3 (Circuit Elimination Property)**.** *If γ_1 and γ_2 are circuits of a matroid and $c \in \gamma_1 \cap \gamma_2$, then there is a circuit γ_3 that is contained in $\gamma_1 \cup \gamma_2 - c$.*

Duality. Matroids have a notion of duality which generalizes orthogonal complements in linear algebra and dual graphs in graph theory.

Let M be a matroid with bases $\mathcal{B}$. Then the set

$$\mathcal{B}^* = \{E - B \,:\, B \text{ is a basis of } M\}$$

is the collection of bases of a matroid $M^* = (E, \mathcal{B}^*)$, called the *dual matroid* M^*. The circuits of the dual matroid M^* are called the *cocircuits* of M.

Deletion, contraction, and minors. We say that an element $e \in E$ is a *loop* of a matroid M if it is contained in no basis; that is, if $\{e\}$ is a dependent set. Dually, e is a *coloop* if it is contained in every basis of M.

The *deletion* $M \backslash e$ of a non-coloop $e \in E$ is the matroid on $E - e$ whose bases are the bases of M that do not contain e. We also call this the *restriction* of M to $E - e$. Dually, the *contraction* M/e of a non-loop $e \in E$ is the matroid on $E - e$ whose bases are the subsets B of $E - e$ such that $B \cup e$ is a basis of M.

It is easy to see that any sequence of deletions and contractions of different elements commutes. We say that a matroid M' is a *minor* of a matroid M if M' is isomorphic to a matroid obtained from M by performing a sequence of deletions and contractions.

Fundamental circuits and cocircuits. Given a basis B and an element $e \in E - B$ there is a unique circuit contained in $B \cup e$, called the *fundamental circuit* of e with respect to B. It is given by

$$\mathrm{Circ}(B, e) = \{x \in E \,:\, B \cup e - x \in \mathcal{B}\}\,.$$

Given a basis B and an element $i \in B$ there is a unique cocircuit disjoint with $B - i$, called the *fundamental cocircuit* of i with respect to B. It is given by

$$\mathrm{Cocirc}(B, i) = \{x \in E \,:\, B \cup x - i \in \mathcal{B}\}\,.$$

Note that the cocircuit $\mathrm{Cocirc}(B,i)$ in M equals the circuit $\mathrm{Circ}(E-B,i)$ in the dual M^*.

Basis activities. Let $<$ be a linear order on the ground set E. For a basis B, define the sets:

$$\begin{aligned} EA(B) &= \{e \in E - B \,:\, \min(\mathrm{Circ}(B,e)) = e\} \\ EP(B) &= \{e \in E - B \,:\, \min(\mathrm{Circ}(B,e)) \neq e\} \end{aligned}$$

The elements of $EA(B)$ and $EP(B)$ are called *externally active* and *externally passive* with respect to B, respectively. Note that $EA(B) \uplus EP(B) = E - B$, where $\uplus$ denotes a disjoint union.

Dually, let

$$\begin{aligned} IA(B) &= \{i \in B \,:\, \min(\mathrm{Cocirc}(B,i)) = i\} \\ IP(B) &= \{i \in B \,:\, \min(\mathrm{Cocirc}(B,i)) \neq i\} \end{aligned}$$

The elements of $IA(B)$ and $IP(B)$ are called *internally active* and *internally passive* with respect to B, respectively. Note that $IA(B) \uplus IP(B) = B$. Also note that the internally active/passive elements with respect to basis B in M are the externally active/passive elements with respect to basis $E - B$ in M^*.

The following elegant result of Tutte (for graphs) and Crapo (for arbitrary matroids) underlies many of the results of [2] and this paper.

Theorem 2.4. [4, 8] Let M be a matroid on the ground set E and let $<$ be a linear order on E.

1. Every subset A of E can be uniquely written in the form $A = B \cup X - Y$ for some basis B, some subset $X \subseteq EA(B)$, and some subset $Y \subseteq IA(B)$. Equivalently, the intervals $[B - IA(B), B \cup EA(B)]$ form a partition of the poset 2^E of subsets of E ordered by inclusion.

2. Every independent set I of E can be uniquely written in the form $I = B - Y$ for some basis B and some subset $Y \subseteq IA(B)$. Equivalently, the intervals $[B - IA(B), B]$ form a partition of the independence complex $IN(M)$.

The external activity complex. Let $\overline{E} = \{\overline{e} \,:\, e \in E\}$ be a "signed" copy of E, and let $[[E]] = E \uplus \overline{E}$. For each $S \subseteq E$ let $\overline{S} := \{\overline{s} \mid s \in S\}$. Our main object of study is the following.

Definition/Theorem 2.5. [2] Let $M = (E, \mathcal{B})$ be a matroid and let $<$ be a linear order on E. M. The *external activity complex* $\mathrm{Act}_<(M)$ is the complex with ground set $[[E]]$ such that

1. The facets are
$$F(B) := B \cup EP(B) \cup \overline{B \cup EA(B)}$$
for every basis $B \in \mathcal{B}$.

2. The minimal non-faces are
$$S(\gamma) = c \cup \overline{\gamma - c}$$
for every circuit γ, where c is the $<$-smallest element of γ.

The complement of the facet $F(B)$ in $[[E]]$ is denoted by $G(B)$ and is given by

$$G(B) = EA(B) \cup \overline{EP(B)}.$$

Las Vergnas's three active orders. Given a matroid $M = (E, \mathcal{B})$ and a total order $<$ on the ground set of M, LasVergnas introduced the following three *active orders*.

Definition/Theorem 2.6. The *external order* $<_{ext}$ on $\mathcal{B}$ is characterized by the following equivalent properties for two bases A and B:

1. $A \leq_{ext} B$,
2. $A \subseteq B \cup EA(B)$,
3. $A \cup EA(A) \subseteq B \cup EA(B)$,
4. B is the lexicographically largest basis contained in $A \cup B$.

This poset is graded with $r(B) = |EA(B)|$. Adding a minimum element turns it into a lattice.

Definition/Theorem 2.7. The *internal order* $<_{int}$ on $\mathcal{B}$ is characterized by the following equivalent properties for two bases A and B:

1. $A \leq_{int} B$,
2. $A - IA(A) \subseteq B$,
3. $A - IA(A) \subseteq B - IA(B)$,
4. A is the lexicographically smallest basis containing $A \cap B$.

This poset is graded with $r(B) = r - |IA(B)|$. Adding a maximum element turns it into a lattice.

The internal and external orders are consistent in the sense that $A \leq_{int} B$ and $B \leq_{ext} A$ imply $A = B$. Therefore the following definition makes sense.

Definition/Theorem 2.8. The *external/internal order* $<_{ext/int}$ is the weakest order which simultaneously extends the external and the internal order. It is characterized by the following equivalent properties for two bases A and B:

1. $A \leq_{ext/int} B$,
2. $IP(A) \cap EP(B) = \emptyset$,
3. $A - IA(A) \cup EA(A) \subseteq B - IA(B) \cup EA(B)$.

This poset is a lattice. It is not necessarily graded.

Note that Theorem 2.6.4 and 2.7.4 imply the following.

Proposition 2.9. The lexicographic order $<_{lex}$ on $\mathcal{B}$ is a linear extension of the three partial orders $<_{int}, <_{ext}$, and $<_{ext/int}$. In other words, any of $A <_{int} B$, $A <_{ext} B$ or $A <_{ext/int} B$ implies $A <_{lex} B$.

2.2 Shellability and the h-vector

Shellability. Shellability is a combinatorial condition on a simplicial complex that allows us to describe its topology easily. A simplicial complex is shellable if it can be built up by introducing one facet at a time, so that whenever we introduce a new facet, its intersection with the previous ones is pure of codimension 1. More precisely:

Definition 2.10. Let Δ be a pure simplicial complex. A *shelling order* is an order of the facets $F_1, \dots F_k$ such for every $i < j$ there exist $k < j$ and $f \in F_j$ such that $F_i \cap F_j \subseteq F_k \cap F_j = F_j - f$. If a shelling order exists, then we call Δ *shellable.*

Given a shelling order and a facet F_j, there is a subset $\mathcal{R}(F_j)$ such that for every $A \subseteq F_j$, we have $A \nsubseteq F_i$ for all $i < j$ if and only if $\mathcal{R}(F_j) \subseteq A$. Equivalently, when we add facet F_j to the complex, the new faces that we introduce are precisely those in the interval $[\mathcal{R}(F_j), F_j]$. The set $\mathcal{R}(F_j)$ is called the *restriction set* of F_j in the shelling.

The f-vector and h-vector. The *f-vector* of a $(d-1)$-dimensional simplicial complex Δ is $(f_0, \dots, f_d)$ where f_i is the number of faces of Δ of size i. The *h-vector* $(h_0, \dots, h_d)$ is an equivalent way of storing this information; it is defined by the relation

$$f_0(x-1)^d + f_1(x-1)^{d-1} + \cdots + f_d(x-1)^0 = h_0x^d + h_1x^{d-1} + \cdots + h_dx^0.$$

This polynomial is also known as the *shelling polynomial* $h_\Delta(x)$, due to the following description of the h-vector for shellable complexes.

Proposition 2.11. [3, Proposition 7.2.3] If $F_1, \dots, F_k$ is a shelling order for a $(d-1)$-dimensional simplicial complex Δ, then

$$h_i := |\{j \,:\, |\mathcal{R}(F_j)| = i\}|.$$

Note that it is not clear a priori that these numbers should be the same for any shelling order.

Understanding the topology of a shellable simplicial complex is easy once we know the last entry of the h-vector, thanks to the following result.

Theorem 2.12. [5, Theorem 12.2(2)] Any geometric realization of a $(d-1)$-dimensional shellable simplicial complex Δ is homotopy equivalent to a wedge of h_d spheres of dimension $d-1$. In particular, if $h_d = 0$, then every geometric realization of Δ is contractible.

An important property for matroids is their shellability:

Theorem 2.13. [3, Theorem 7.3.3] The lexicographic order $<_{lex}$ on the bases of a matroid M gives a shelling order of the independence complex $IN(M)$.

3 Example

Before proving our theorems, we illustrate them in an example. Consider the graphical matroid given by the graph of Figure 1. Its bases are all the 3-subsets of [5] except $\{1,2,3\}$ and $\{1,4,5\}$. Under the standard order $1 < 2 < 3 < 4 < 5$ on the ground set, Table 1 records the basis activity of the various bases.

The resulting internal, external, and external/internal orders $<_{int}, <_{ext}, <_{ext/int}$ are shown in Figure 2. By Theorems 2.6, 2.7, and 2.8, these three orders are isomorphic to the three families

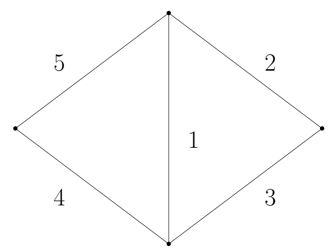

Figure 1: A graphical matroid.

B	$EP(B)$	$EA(B)$	$IP(B)$	$IA(B)$
124	35	$\emptyset$	$\emptyset$	124
125	45	$\emptyset$	5	12
134	25	$\emptyset$	3	14
135	24	$\emptyset$	35	1
234	5	1	23	4
235	4	1	235	$\emptyset$
245	3	1	45	2
345	$\emptyset$	12	345	$\emptyset$

Table 1: The bases B together with their sets of externally passive, externally active, internally passive, and internally active elements.

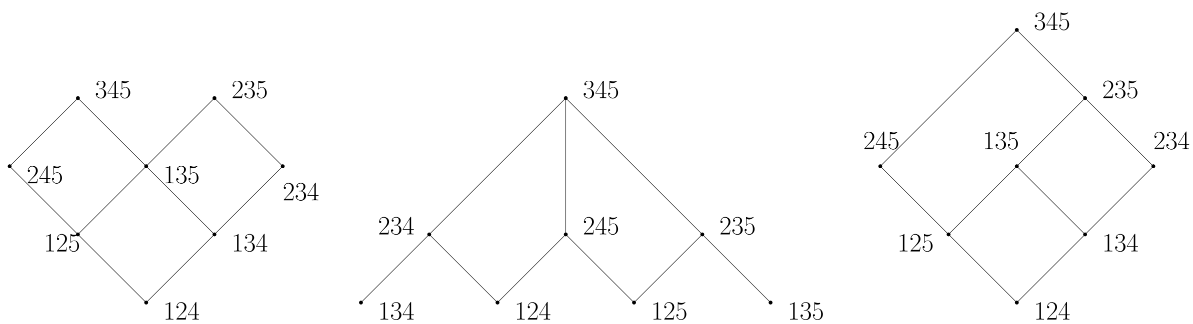

Figure 2: The active orders $<_{int}, <_{ext}$, and $<_{ext/int}$, respectively.

of sets $\{B \cup EA(B) : B \text{ basis}\}$, $\{B - IA(B) : B \text{ basis}\}$, and $\{B \cup EA(B) - IA(B) : B \text{ basis}\}$, partially ordered by containment.

Table 1 lists the bases in lexicographic order $<_{lex}$, and this is a shelling order for the independence complex $IN(M)$ by Theorem 2.13. The restriction set for each basis B is $\mathcal{R}(B) = IP(B)$. For example, when we add facet 134 in the third step of the shelling, this means that the new faces that appear are the four sets in the interval $[\mathcal{R}(134), 134] = [3, 134]$; that is, faces $3, 13, 34$, and 134.

Our goal is to shell the external activity complex $\text{Act}_<(M)$ whose facets, listed in Table 2, are the sets $F(B) = B \cup EP(B) \cup \overline{B \cup EA(B)}$. Since $\overline{1}, 3, 4$, and 5 are in all facets of $\text{Act}_<(M)$, we remove them, and shell the resulting *reduced external activity complex* $\overline{Act}_<(M)$. Our main result, Theorem 1.1, states that any linear extension of the external/internal order $<_{ext/int}$ gives a shelling order for this complex. For example, we may again consider the lexicographic order, which is indeed a linear extension of $<_{ext/int}$.

B	$F(B)$	$\overline{F(B)}$	$\mathcal{R}(F(B))$
124	$12345\overline{124}$	$12\overline{24}$	$\emptyset$
125	$12345\overline{125}$	$12\overline{25}$	$\overline{5}$
134	$12345\overline{134}$	$12\overline{34}$	$\overline{3}$
135	$12345\overline{135}$	$12\overline{35}$	$\overline{35}$
234	$2345\overline{1234}$	$2\overline{234}$	$\overline{23}$
235	$2345\overline{1235}$	$2\overline{235}$	$\overline{235}$
245	$2345\overline{1245}$	$2\overline{245}$	$\overline{45}$
345	$345\overline{12345}$	$\overline{2345}$	$\overline{345}$

Table 2: The bases B of M, the corresponding facets $F(B)$ and $\overline{F(B)}$ of $\mathrm{Act}_<(M)$ and $\overline{\mathrm{Act}}_<(M)$, and their (shared) restriction set $\mathcal{R}(F(B))$ in the shelling.

For each basis B, Table 2 lists the corresponding facet $F(B)$ of $\mathrm{Act}_<(M)$, the corresponding facet $\overline{F(B)}$ of $\overline{\mathrm{Act}}_<(M)$, and the restriction set of the facet $F(B)$ in the shelling. This restriction set is $\mathcal{R}(F(B)) = \overline{IP(B)}$. For example, when we add facet $12\overline{34}$ to the complex $\overline{Act}_<(M)$ in the third step of the shelling, the new faces that appear are the eight sets in the interval $[\mathcal{R}(12\overline{34}), 12\overline{34}] = [\overline{3}, 12\overline{34}]$.

Notice that we can embed $IN(M) \longrightarrow \overline{Act}_<(M)$ by sending $1 \to 1, 2 \to \overline{2}, 3 \to \overline{3}, 4 \to \overline{4}, 5 \to \overline{5}$. The latter complex has the same h-vector and is contractible. Therefore, it is no coincidence that the shellings of $IN(M)$ and $Act_<(M)$ are related. In fact, we will prove that any shelling order for $Act_<(M)$ is a shelling order for $IN(M)$. Theorem 1.1 then gives:

$$\text{any linear extension of } <_{ext/int} \text{ is a shelling order for } IN(M) \text{ and } Act_<(M). \tag{1}$$

We conclude this section with two examples showing that the linear extensions of the internal and external orders $<_{int}$ and of $<_{ext}$ are not necessarily shelling orders for $Act_<(M)$.

Example 3.1. Consider any linear extension of $<_{ext}$ starting with 124 and 135 in that order, such as:

$$124, 135, 125, 134, 234, 235, 245, 345.$$

This is not a shelling order for $IN(M)$ because the second facet 135 intersects the first facet 124 in codimension 2. By Corollary 4.3 (or directly by inspection), this is not a shelling order for $Act_<(M)$ either. Therefore:

$$\text{a linear extension of } <_{ext} \text{ need not be a shelling order for } IN(M) \text{ or for } Act_<(M). \tag{2}$$

Example 3.2. Consider the following linear extension of $<_{int}$:

$$124, 125, 134, 135, 245, 345, 234, 235$$

which gives the following order on the facets:

$$12\overline{24},\ 12\overline{25},\ 12\overline{34},\ 12\overline{35},\ 2\overline{245},\ \overline{2345},\ 2\overline{234},\ 2\overline{235},$$

This is a shelling of $IN(M)$ by Theorem 1.2. However, it is not a sheling of $\mathrm{Act}_<(M)$ and $\overline{\mathrm{Act}}_<(M)$. To see this, suppose we introduce the facets of $\overline{Act}_<(M)$ in the order above. When we introduce the sixth facet $\overline{2345}$ we introduce two new minimal faces: $\overline{23}$ and $\overline{345}$; so this is not a shelling order for $Act_<(M)$. Hence

$$\text{a linear extension of } <_{int} \text{ is a shelling order for } IN(M)\text{, but not necessarily for } Act_<(M). \quad (3)$$

In summary, combining (1), (2), and (3), we see that the hypotheses of Theorems 1.1 and 1.2 are as strong as possible in the context of LasVergnas's active orders.

4 Shellability of the external activity complex

In this section we prove our main result, which states that the external activity complex is shellable. We begin by proving two technical lemmas.

Lemma 4.1. Let M be a matroid on an ordered ground set, and let A, C be bases of M. There exist $c \in EP(A) \cap C$ and $a < c$ such that $C - c \cup a$ is a basis if and only if $A \not\geq_{ext/int} C$ in LasVergnas's external/internal order.

Proof. Given $c \in C$, we can find an element $a < c$ with $C - c \cup a \in \mathcal{B}$ if and only if $c \in IP(C)$. To find such an element c with the additional condition that $c \in EP(A)$, we need $IP(C) \cap EP(A) \neq \emptyset$; this is equivalent to $A \not\geq_{ext/int} C$ in LasVergnas's external/internal order by Theorem 2.8.2. □

A total order $<$ on the set $\mathcal{B}$ of bases of M induces an order on the set of facets $\{F(B) : B \in \mathcal{B}\}$ of the external activity complex $\mathrm{Act}_<(M)$. We now characterize the shelling orders on $\mathrm{Act}_<(M)$.

Lemma 4.2. Let $\mathcal{B}$ be the set of bases of a matroid M. A total order $<$ on the set $\mathcal{B}$ induces a shelling of the external activity complex $\mathrm{Act}_<(M)$ if and only if for any bases $A < C$ there exists a basis $B < C$ such that

(a) $B = X \cup b$ and $C = X \cup c$ for some $b \neq c$.

(b) $c \notin A$ and $c \in EA(B)$ if and only if $c \in EA(A)$.

(c) For any $d \notin B \cup C = X \cup b \cup c$ we have $d \in EA(B)$ if and only if $d \in EA(C)$

Proof. By definition, $<$ induces a shelling order if for every $A < C$ there exist $B < C$ and $c^\pm \in F(C)$ (where $c^\pm$ equals c or $\overline{c}$ for some $c \in E$) such that

$$F(A) \cap F(C) \subset F(B) \cap F(C) = F(C) - c^\pm.$$

Recalling that $G(D) = EA(D) \cup \overline{EP(D)}$ is the complement of $F(D)$ in $[[E]]$ for each basis D, this is equivalent to

$$G(A) \cup G(C) \supset G(B) \cup G(C) = G(C) \cup c^\pm.$$

Define the support of $S \subset [[E]]$ to be $\mathrm{supp}(S) = \{i \in E : i \in S \text{ or } \overline{i} \in S\}$. Notice that we have $\mathrm{supp}(G(D)) = E - D$ for any basis D. Then

$$|E| - |B \cap C| = |\mathrm{supp}(G(B) \cup G(C))| = |\mathrm{supp}(G(C) \cup c^\pm)| = |E| - r + 1.$$

where r is the rank of the matroid. This implies (a).

If (c) was not satisfied for some $d \notin B \cup C$, we would find both d and $\overline{d}$ in $G(B) \cup G(C) = G(C) \cup c$, a contradiction. Finally, $c^{\pm}$ is in $G(A)$ and $G(B)$, which implies (b).

The converse follows by a very similar argument. $\square$

Corollary 4.3. If a total order $<$ on $\mathcal{B}$ induces a shelling of the external activity complex $\mathrm{Act}_{<}(M)$, then it also induces a shelling of the independence complex $IN(M)$.

Proof. Let $A < C$ and assume that $B < C$ satisfy conditions (a), (b), and (c) of Lemma 4.2. Since $\mathrm{supp}(G(D)) = E - D$ for every basis D, the containment $G(A) \cup G(C) \supset G(B) \cup G(C)$ gives $E - (A \cap C) \supset E - (B \cap C)$, which implies $A \cap C \subset B \cap C = X = C - c$. Hence the total order $<$ induces a shelling order of $IN(M)$. $\square$

Now we are ready to prove our main theorem.

Theorem 1.1. Let $M = (E, \mathcal{B})$ be a matroid, and let $<$ be a linear order on the ground set E. Any linear extension of LasVergnas's external/internal order $<_{ext/int}$ of $\mathcal{B}$ induces a shelling of the external activity complex $\mathrm{Act}_{<}(M)$.

Proof. We use the characterization of Lemma 4.2. Consider bases $A < C$; we will find the desired basis in two steps. We construct a basis B and, if necessary, a second basis B', and we will show that one of them satisfies the conditions (a),(b),(c) of Lemma 4.2.

Step 1. Since $A \ngeq_{ext/int} C$, we first use Lemma 4.1 to find $c \in EP(A) \cap C$ and a minimal element $b < c$ such that

$$B = X \cup b$$

is a basis, where $X = C - c$. The minimality of b implies that b is minimum in $\mathrm{Cocirc}(B, b)$, so $b \in IA(B)$. Therefore $B \backslash IA(B) \subseteq X \subseteq C$. Theorem 2.7 then implies that $B <_{int} C$, which in turn gives $B <_{ext/int} C$, and hence $B < C$.

Property (a) is clearly satisfied. By construction $c \notin A$ and $c \in EP(A)$. Since $b < c$ is in $\mathrm{Circ}(B, c)$, we have $c \in EP(B)$. Therefore (b) is also satisfied. Property (c) does not always hold; let us analyze how it can fail, and adjust B accordingly if necessary.

Suppose (c) fails for an element $d \notin B \cup C$; call such an element a $\{B, C\}$ *external disagreement.* This means that d is minimum in one of the fundamental circuits $\beta = \mathrm{Circ}(B, d)$ and $\gamma = \mathrm{Circ}(C, d)$ but not in the other one.

Since they have different minima, we have $\beta \neq \gamma$; so using circuit elimination, we can find a circuit $\alpha \subseteq \beta \cup \gamma - d$. This circuit must contain b and c, or else it would be contained in basis B or C. This implies that

$$b, c \in \alpha, \qquad b, d \in \beta, \qquad c, d \in \gamma.$$

It follows that $D = X \cup d = (B \cup d) - b$ is a basis. By the uniqueness of fundamental circuits, we must have

$$\alpha = \mathrm{Circ}(B, c) = \mathrm{Circ}(C, b), \quad \beta = \mathrm{Circ}(B, d) = \mathrm{Circ}(D, b), \quad \gamma = \mathrm{Circ}(C, d) = \mathrm{Circ}(D, c).$$

Taking into account that $b < c$, we consider three cases:

- **1.** $b < c < d$**:** Since $b \in \mathrm{Circ}(B, d) = \beta$ and $c \in \mathrm{Circ}(C, d) = \gamma$, d is minimum in neither β nor γ, a contradiction.

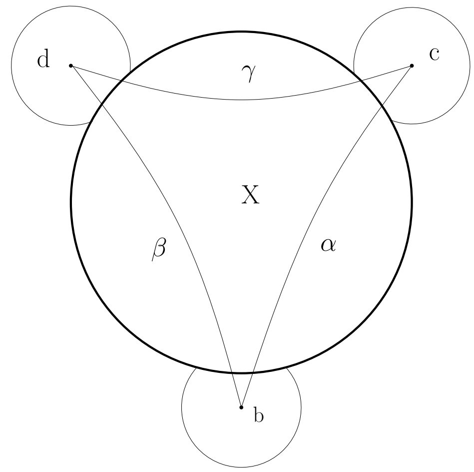

Figure 3: The bases $B = X \cup b$, $C = X \cup c$, and $D = X \cup d$ and the fundamental circuits β, γ, α.

- **2.** $d < b < c$**:** The minimality of b implies that $X \cup d = D$ is not a basis, a contradiction.
- **3.** $b < d < c$**:** Since d is not minimum in $\text{Circ}(B, d) = \beta \ni b$, we have $d \in EP(B)$; so d is a $\{B, C\}$ external disagreement if and only if $d \in EA(C)$.

We conclude that, under the above hypotheses,

$$d \text{ is a } \{B, C\} \text{ external disagreement} \iff X \cup d \text{ is a basis, } b < d < c, \text{ and } d \in EA(C). \tag{4}$$

If there are no $\{B, C\}$ external disagreements, B is our desired basis. Otherwise, proceed as follows.

Step 2. Define the basis

$$B' = X \cup b'$$

where b' is the largest $\{B, C\}$ external disagreement. We have $b < b' < c$ and $b' \in EA(C)$. It follows that $B' \subset C \cup EA(C)$, so $B' <_{ext} C$ by Theorem 2.6. This implies that $B' <_{ext/int} C$, which in turn gives $B' < C$. Now we claim that B' satisfies conditions (a),(b),(c) of Lemma 4.2.

Property (a) is clearly satisfied. By construction $c \notin A$ and $c \in EP(A)$. Since $b' < c$ is in $\text{Circ}(B', c)$, we have $c \in EP(B')$, so (b) holds. To show (c), assume contrariwise that $d' \notin X \cup b' \cup c$ is a $\{B', C\}$ external disagreement; that is, it is minimum in one of the fundamental circuits $\beta' = \text{Circ}(B', d')$ and $\gamma' = \text{Circ}(C, d')$ but not in the other.

As in Step 1, $D' = X \cup d'$ must be a basis, and we have circuits

$$\alpha' = \text{Circ}(B', c) = \text{Circ}(C, b'), \quad \beta' = \text{Circ}(B', d') = \text{Circ}(D', b'), \quad \gamma' = \text{Circ}(C, d') = \text{Circ}(D', c).$$

with

$$b', c \in \alpha', \qquad b', d' \in \beta', \qquad c, d' \in \gamma'.$$

Once again, in view of $b' < c$, we consider three cases:

- **Case 1** $b' < c < d'$**:** Since $b' \in \text{Circ}(B, d') = \beta'$ and $c \in \text{Circ}(C, d') = \gamma$, d' is minimum in neither β nor γ, a contradiction.

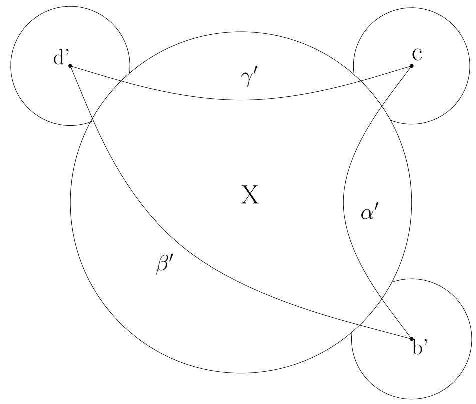

Figure 4: The bases $B' = X \cup b'$, $C = X \cup c$, and $D' = X \cup d'$ and the fundamental circuits β', γ', α'.

- **Case 2** $d' < b' < c$**:** If $d' \in EA(B')$ then $d' = \min \beta'$. Since $b' \in EA(C)$, we have $b' = \min \alpha'$. Because they have different minima, we have $\beta' \neq \alpha'$, so we can use circuit elimination to find a circuit $\gamma'' \subseteq (\alpha' \cup \beta') - b'$. Again, that circuit must contain c and d' or else it would be contained in C or D'. Therefore, by the uniqueness of fundamental circuits, $\gamma'' = \gamma'$. Now, since $\gamma' \subseteq (\alpha' \cup \beta') - b'$ and $d' \in \gamma'$, we have $d' = \min \gamma'$ and $d' \in EA(C)$.

  Similarly, if $d' \in EA(C)$ then $d' = \min \gamma'$. Since $b' \in EA(C)$, we have $b' = \min \alpha'$. As above, we can conlcude that $\beta' \subseteq (\alpha' \cup \gamma') - c$ and $d' \in \beta'$, we have $d' = \min \beta'$ and $d' \in EA(B')$.

  In either case, we get a contradiction.

- **Case 3** $b' < d' < c$ Since d' is not minimum in $\beta' = \mathrm{Circ}(B, d') \ni b'$, if d' is a $\{B', C\}$ external disagreement, it must be minimum in $\gamma = \mathrm{Circ}(C, d')$; that is, $d' \in EA(C)$. We have $b < b' < d' < c$, and $X \cup d'$ is a basis. Therefore, recalling (4), d' is also a $\{B, C\}$ external disagreement, contradicting the maximality of b'.

In conclusion, there are no $\{B', C\}$ external disagreements, and property (c) holds. Therefore the basis B' has all required properties. □

Corollary 4.4. Any linear extension of the external/internal order $<_{ext/int}$ gives a shelling order for the independence complex $IN(M)$.

Proof. This follows from Theorem 1.1 and Corollary 4.3. □

In fact, we now prove a stronger result. We begin with a useful lemma.

Lemma 4.5. Let I be an independent set of M and let C be any basis that contains I. If B is the lexicographically smallest basis that contains I then $B \leq_{int} C$.

Proof. By Theorem 2.7.4, we need to show that B is the lexicographically smallest basis that contains $B \cap C$. To do so, assume there is a basis $A \supseteq B \cap C$ with $A <_{lex} B$ Then $A \supseteq B \cap C \supseteq I$, contradicting the minimality of B. □

Theorem 1.2. Any linear extension of the internal order $<_{int}$ gives a shelling order of the independence complex $IN(M)$.

Proof. Let $<$ be any linear extension of $<_{int}$, and let $A < C$ be bases, so $A \not>_{int} C$. We claim that there exists $B <_{int} C$ (and hence $B < C$) such that $A \cap C \subseteq B \cap C = C - c$ for some c in C. This will prove the desired result.

To show this, let D be the lexicographically smallest basis that contains $A \cap C$. Notice that $D \neq C$ because $A \not>_{int} C$, using Theorem 2.7.4. Let d be smallest element in $D - C$ and let c be any element of $C - D$ such that $C' = C - c \cup d$ is a basis. Also notice that $D <_{int} C$ by Lemma 4.5; and since $<_{lex}$ is a linear extension of $<_{int}$, we have $D <_{lex} C$. This gives $d = \min(D - C) < \min(C - D) \leq c$, and therefore $C' <_{lex} C$.

Put $X = C - c$ and let B be the lexicographically smallest basis that contains X. Since C' contains X, $B \leq_{lex} C' <_{lex} C$, so $B \neq C$. Therefore $B <_{int} C$ by Lemma 4.5. Also note that, since $c \notin D \supset A \cap C$ and $c \in C$, we must have $c \notin A$. This gives $A \cap C \subseteq C - c = X$, and therefore $A \cap C \subseteq B \cap C = X$. It follows that B satisfies the desired properties. □

5 The h-vector

We now describe the restriction sets for the shellings of Theorem 1.1.

Proposition 5.1. Let $<$ be any linear extension of $<_{ext/int}$, and regard it as a shelling order for $IN(M)$. Then the restriction set of each facet C (which is a basis of M) is $IP(C)$.

Proof. We need to show $IP(C)$ is the minimum subset of C which is not a subset of a basis $B < C$.

To show that $IP(C)$ indeed has this property, assume that if $IP(C) \subseteq B$. Then by Theorem 2.7.2, we have $C \leq_{int} B$ and hence $C \leq B$, as desired.

To show minimality, let $U \subsetneq IP(C)$. By Theorem 2.4.2 we can find a basis A such that $A - IA(A) \subseteq U \subseteq A$. This gives $A - IA(A) \subseteq U \subsetneq C - IA(C)$, which in light of Theorem 2.7.3 gives $A <_{int} C$, and hence $A < C$. Therefore U is a subset of A with $A < C$, as desired. □

Proposition 5.2. Let $<$ be any linear extension of $<_{ext/int}$, and regard it as a shelling order for $\mathrm{Act}_<(M)$. Then the restriction set of each facet $F(C)$ (where C is a basis of M) is $\overline{IP(C)}$.

Proof. We need to show $\overline{IP(C)}$ is the minimum subset of $F(C)$ which is not a subset of $F(B)$ for any basis $B < C$.

To show $\overline{IP(C)}$ does have this property, assume that $\overline{IP(C)} \subseteq F(B) = B \cup EP(B) \cup \overline{B \cup EA(B)}$ for some basis B. Then $IP(C) \subset B \cup EA(B)$, so $IP(C) \cap EP(B) = \emptyset$. By Theorem 2.8.2, $C <_{ext/int} B$ so $C < B$, as desired.

To show minimality, let $\overline{U} \subsetneq \overline{IP(C)}$, so $U \subsetneq IP(C)$. By Proposition 5.1, U is contained in a basis $A < C$, and hence $\overline{U}$ is contained in $F(A)$ for that basis, as desired. □

As an immediate consequence, we obtain our main enumerative result.

Theorem 1.3. The h-vector of $\mathrm{Act}_<(M)$ equals the h-vector of M.

Proof. This follows from the previous two results, in light of Proposition 2.11. □

6 Topology

The external activity complex $\mathrm{Act}_<(M)$ is a cone; for example, it is easy to see that every facet contains $\overline{\min E}$ and $\max E$. Therefore $\mathrm{Act}_<(M)$ is trivially contractible. It is more interesting to study the topology of the *reduced external activity complex* $\overline{Act}_<(M)$, obtained by removing all cone points of $\mathrm{Act}_<(M)$. It turns out that Corollary 1.3 gives us enough information to describe it. First we need a few technical lemmas.

Define a *loop* of a simplicial complex Δ to be an element l of the ground set such that $\{l\}$ is not a face of Δ. Say an element e of a matroid M is *absolutely externally active* if it is externally active with respect to every basis not containing it, or *absolutely externally passive* if it is externally passive with respect to every basis not containing it. Let $AEA(M)$ and $AEP(M)$ be the respective sets of elements, and call the elements of $AE(M) = AEA(M) \cup AEP(M)$ *externally absolute*.

Lemma 6.1. The set of cone points of $Act_<(M)$ is $AEP(M) \cup \overline{AEA(M)}$. The ground set of $\overline{Act}_<(M)$ is $\{e : e \notin AEP(M)\} \cup \{\overline{e} : e \notin AEA(M)\}$, and this simplicial complex has no loops.

Proof. The first two statements are clear from the definitions. For the last one, if $e \notin AEP(M)$, then we can find a basis B with respect to which e is externally active, so $\{\overline{e}\} \subset F(B)$ is a face of $\overline{\mathrm{Act}}_<(M)$. Similarly, if $e \notin AEA(M)$, then we can find a basis B with respect to which e is externally passive, so $\{e\} \subset F(B)$ is a face of $\overline{\mathrm{Act}}_<(M)$. □

Lemma 6.2. Let $M = (E, \mathcal{B})$ be a matroid. Every element $e \in E$ is externally absolute if and only if the circuits of M are pairwise disjoint.

Proof. The backward direction is a straightforward consequence of the definitions. To prove the forward direction, we proceed by contradiction. Assume that every element of M is externally absolute, and that we have two circuits γ_1 and γ_2 with $\gamma_1 \cap \gamma_2 \neq \emptyset$ whose minimal elements are c_1 and c_2, respectively. Consider two cases.

1. If $c_1 = c_2$ then perform circuit elimination to get $\gamma_3 \subset \gamma_1 \cup \gamma_2 - c_1$. Let c_3 be the minimal element of γ_3; without loss of generality assume $c_3 \in \gamma_1$. Then c_3 is externally active for some basis, as testified by γ_3, and it is externally passive for another basis, as testified by γ_1. Hence c_3 is not absolute, a contradiction

2. If $c_1 \neq c_2$ and $c \in \gamma_1 \cap \gamma_2$, then perform circuit elimination with c to get a circuit $\gamma_3 \subset \gamma_1 \cup \gamma_2 - c$. Let c_3 be the minimal element of γ_3; assume $c_3 \in \gamma_1$. If $c_3 = c_1$, then case 1 applies to circuits γ_1 and γ_3, and we get a contradiction. Otherwise, we must have $c_1 < c_3$ since $c_1 = \min \gamma_1$. Therefore c_3 is externally active for some basis, as testified by γ_3, and externally passive for another basis, as testified by γ_1, a contradiction. □

Lemma 6.3. If a matroid is the disjoint union of circuits, then $\overline{\mathrm{Act}}_<(M) \cong IN(M)$. Otherwise, $\overline{\mathrm{Act}}_<(M)$ has a proper subcomplex which is isomorphic to $IN(M)$. The embedding may be chosen so that the image of facet B of $IN(M)$ is a subset of the facet $F(B)$ of $\overline{\mathrm{Act}}_<(M)$.

Proof. For every $e \in E$ let $e' = e$ if e is absolutely externally active, and $e' = \overline{e}$ otherwise. The set $E' = \{e' : e \in E\}$ is a subset of the vertices of $\overline{\mathrm{Act}}_<(M)$ by Lemma 6.1. For every basis B of M the set $B' = \{b' : b \in B\}$ is a subset of $F(B)$, and hence a face of $\overline{\mathrm{Act}}_<(M)$. This gives the desired embedding of $IN(M)$ in $\overline{\mathrm{Act}}_<(M)$.

If M is the disjoint union of circuits, then E' equals the ground set of $\overline{\mathrm{Act}}_<(M)$, and B' equals $F(B) \cap E'$ for all bases B, so this embedding is actually an isomorphism.

If M is not the disjoint union of circuits, by Lemma 6.1, E' is a proper subset of the ground set of $\overline{\mathrm{Act}}_<(M)$, so the embedding of $IN(M)$ is a proper subcomplex of $\overline{\mathrm{Act}}_<(M)$. $\square$

Lemma 6.4. If a matroid M of rank r is the disjoint union of circuits, then the independence complex $IN(M)$ is homeomorphic to an $(r-1)$-sphere.

Proof. If M is a single circuit (necessarily of size $r+1$), then $IN(M)$ is the boundary of an r-simplex, and hence an $(r-1)$-sphere.

If M is the disjoint union of circuits $\gamma_1, \ldots, \gamma_k$ then $IN(M)$ is the join of $IN(\gamma_1), \ldots, IN(\gamma_k)$; that is, $IN(M) = IN(\gamma_1) \star \cdots \star IN(\gamma_k) = \{A_1 \cup \cdots \cup A_k \, : \, A_i \in IN(\gamma_i) \text{ for } 1 \leq i \leq k\}$. The result then follows from the fact that the join of two spheres $\mathbb{S}^k$ and $\mathbb{S}^l$ is homeomorphic to the sphere $\mathbb{S}^{k+l+1}$. [5, Chapter 2.2.2] $\square$

The matroids with pairwise disjoint cycles have a nice characterization in terms of excluded minors.

Lemma 6.5. A matroid M contains two circuits with non empty intersection if and only if $U_{3,1}$ is a minor of M.

Proof. First suppose that M contains two intersecting circuits γ and δ which intersect at e. Let $c \in \gamma - \delta$ and $d \in \delta - \gamma$. Restricting to $\gamma \cup \delta$ and then contracting every element except for c, d, and e, we obtain $U_{3,1}$ as a minor.

To show the converse consider any matroid N and an element $e \in E$. Notice that every circuit of $N \backslash e$ is a circuit of N; and if γ is a circuit of N, then either γ or $\gamma \cup e$ is a circuit of N. It follows that if either $N \backslash e$ or N/e have two overlapping circuits, so does N. Since $U_{3,1}$ has two overlapping circuits, so does every matroid containing it as a minor. $\square$

Now we are ready to prove our main topological result.

Theorem 1.4. Let M be a matroid and $<$ be a linear order on its ground set. The reduced external activity complex $\overline{Act}_<(M)$ is contractible if M contains $U_{3,1}$ as a minor, and a sphere otherwise.

Proof. Notice that if M has a coloop c, then both c and $\bar{c}$ are cone points of $\mathrm{Act}_<(M)$, and are invisible in $\overline{Act}_<(M)$. Therefore we may assume that M is coloop free.

Let r be the rank of M, and let $d = \dim(\overline{Act}_<(M)) = \dim(Act_<(M)) - |AE(M)| = n + r - 1 - |AE(M)|$. We consider two cases.

1. If M is not the disjoint union of circuits, $|AE(M)| < n$ by Lemma 6.2, so $d > r - 1$. Clearly $h_d(\overline{Act}_<(M)) = h_d(Act_<(M))$, Theorem 1.3 gives $h_d(Act_<(M)) = h_d(IN(M))$, and since $IN(M)$ is $(r-1)$-dimensional, $h_d(IN(M)) = 0$. Therefore, by Theorem 2.12, $\overline{Act}_<(M)$ is contractible.

2. If M is the disjoint union of circuits, then $\overline{Act}_<(M) \cong IN(M)$ is a sphere invoking Lemmas 6.3 and 6.4.

The result follows from Lemma 6.5. $\square$

We conclude that the simplicial complex $\overline{\mathrm{Act}}_<(M)$ is a model for a matroid M which is topologically simpler than the "usual" model $IN(M)$.

7 Questions

- There should be "affine" analogs of the results of this paper. Geometrically, they should correspond to taking the closure of an affine subspace L of $\mathbb{A}^n$ in $(\mathbb{P}^1)^n$, as opposed to a linear subspace, as explained in [2]. To a *morphism of matroids* $M \to M'$, one may associate an external activity complex $\mathrm{Act}_<(M \to M')$ [2] and active orders $<_{int}, <_{ext}, <_{ext/int}$ [6]. The analogous foundational results, such as Theorems 2.4, 2.6, 2.7, 2.8 hold there as well. [1, 6] Do our main theorems hold in that more general setting?

- Even though $\mathrm{Act}_<(M)$ only pays attention to the external activities of the bases of M, it is the external/internal order $<_{ext/int}$ which plays a crucial role in its shelling. This makes the following question from [2] even more natural: is $\mathrm{Act}_<(M)$ part of a larger (and well-behaved) simplicial complex which simultaneously involves the internal and external activities of the bases of M? Ideally we would like it to come from a natural geometric construction.

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