when do two planted graphs have the same cotransversal matroid?

federico ardila[*] amanda ruiz[†]

abstract

Cotransversal matroids are a family of matroids which arise from planted graphs. We describe when two planted graphs give rise to the same cotransversal matroid. We define a family of local moves on a planted graph which preserve the matroid. We prove that if two planted graphs give the same cotransversal matroid, then they can be obtained from each other by a series of these local moves.

1 introduction

Cotransversal matroids are a family of matroids which arise from planted graphs. The goal of this paper is to describe when two planted graphs give rise to the same cotransversal matroid.

In Sections 2 and 3 we define the operations of *swapping* and *maximizing* on a planted graph (G, B), and prove that these operations preserve the cotransversal matroid $L(G, B)$. Conversely, in Section 5 we show that any two maximal planted graph presentations of the same cotransversal matroid can be obtained from each other by a series of swaps.

The results in this paper are motivated by and analogous to Whitney's 2-Isomorphism Theorem for graphical matroids [4, 6]. Whitney's theorem defines three operations on an undirected graph G which do not affect the graphical matroid $M(G)$, and proves that any two graphs with the same graphical matroid can be obtained from each other by a series of these operations.

Throughout the paper we will assume some basic knowledge of matroid theory. For more background information we refer the reader to [4].

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[*]Department of Mathematics, San Francisco State University, federico@math.sfsu.edu.
Supported by National Science Foundation grant DMS-0801075.
[†]Department of Mathematics, San Francisco State University, alruiz@sfsu.edu.
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2 swapping

In this section we introduce the *swap* operation on a planted graph (G, B), and show that it preserves the cotransversal matroid $L(G, B)$. We start with some preliminaries.

Definition 2.1. *A* matroid $(E, \mathcal{B})$ *consists of a finite set* E *and a nonempty family* $\mathcal{B}$ *of subsets of* E*, called* bases *such that:*

> *If* $B_a, B_b \in \mathcal{B}$ *and* $x \in B_a - B_b$*, then there exists* $y \in B_b - B_a$ *such that* $(B_a - x) \cup y \in \mathcal{B}$*.*

In a directed graph with no parallel edges, we denote the edge from i to j by e_{ij}. A *sink* in G is a vertex with no outgoing edges, and a *routing* is a set of vertex-disjoint paths.

Definition 2.2. *A* planted graph (G, B) *on* V *is a directed graph* G *on the vertex set* V *having no parallel edges, together with a specified set of sinks* $B \subseteq V$*.*

Theorem 2.3. *[4] Given a planted graph* (G, B) *on* V*, there is a matroid* $L(G, B)$ *on* V *whose bases are the sets of* $|B|$ *vertices that can be routed to* B *through vertex-disjoint directed paths. Any matroid* M *that arises in this way is called* cotransversal*, and a planted graph giving rise to it is called a* presentation *of* M*.*

Definition 2.4. *Let* (G, B) *be a planted graph, and let* $i \notin B, j \in B$ *be such that* $e_{ij} \in G$*. The* swap operation $\mathbf{swap(i,j)}$ *turns* (G, B) *into the planted graph* $(G, B)_{i \to j} = (G', B')$ *by*

- *replacing* $e_{ij} \in G$ *with* $e_{ji} \in G'$*,*
- *replacing every other edge of the form* e_{ik} *in* G *with* $e_{jk} \in G'$*, and*
- *replacing the sink* $j \in B$ *with the new sink* $i \in B'$*.*

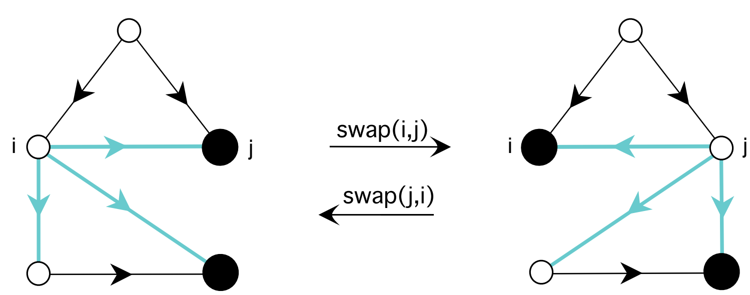

Figure 1: The operation $\mathbf{swap(i,j)}$.

Figure 1 illustrates the operation $\mathbf{swap(i,j)}$; the set B is represented by large, black vertices. Notice that $\mathbf{swap(j,i)}$ is a two-sided inverse of $\mathbf{swap(i,j)}$.

Proposition 2.5. *If* (G, B) *is a planted graph, and* $i \notin B, j \in B$ *are such that* $e_{ij} \in G$*, then* $L((G, B)_{i \to j}) = L(G, B)$.

Proof. Since $\mathbf{swap(i, j)}$ is invertible, it suffices to show that any set of vertices which could be routed to B in (G, B) can be routed to B' in $(G, B)_{i \to j} = (G', B')$.

Let A be a basis of $L(G, B)$, and consider a routing R from A to B. Denote the path in R which goes from a to b by p_{ab}. Let v be the vertex of A which gets routed to j. We consider the following three cases: (i) v is routed through i to get to j, (ii) v is routed to j without going through i, and i is not in any other route of R, and (iii) v is routed to j without going through i, and i is in some other route of R.

(i) Since e_{ij} is in G, we can assume that R uses the path $p_{vj} = (v, \ldots, i, j)$ from v to j. As a result of the operation $\mathbf{swap(i, j)}$ we have $B' = B - j \cup i$. The operation $\mathbf{swap(i, j)}$ does not affect the path from v to i, or any other paths in R. We can replace the path p_{vj} in R with the path $p'_{vi} = p_{vj} - e_{ij}$ of G', and let the other paths of the routing stay the same. Therefore $A \in L(G', B')$.

(ii) Since i is not on the route from v to j, no edges along the path p_{vj} are affected by the swap, so v still has this path to j in G'. Also $e_{ji} \in (G', B')$, so the path $p'_{vi} = p_{vj} \cup e_{ji} \in (G', B')$ routes v to i and doesn't intersect the other paths of the routing. We obtain that $A \in L(G', B')$.

(iii) Let w be the vertex of A which is routed through i to some sink $b \in B$, $b \neq j$, as shown in Figure 2. As a result of the operation $\mathbf{swap(i, j)}$, the path $p_{wb} \in (G, B)$ gets blocked at the edge e_{ik}. We can use the truncated path $p'_{wi} = (w, \ldots, i) \in (G', B')$, as a route from w to $i \in B'$. To complete a routing we need a path leaving $v \in A$ and arriving at $b \in B'$. The path p_{vj} in G is unaffected in G', and $e_{jk} \in (G', B')$ since $e_{ik} \in (G, B)$. So we can use the old path p_{vj} and the new bridge $e_{jk} \in G'$ to pick up the old path from k to b; this does not intersect any other path in the routing R. It follows that $A \in L(G', B')$. $\square$

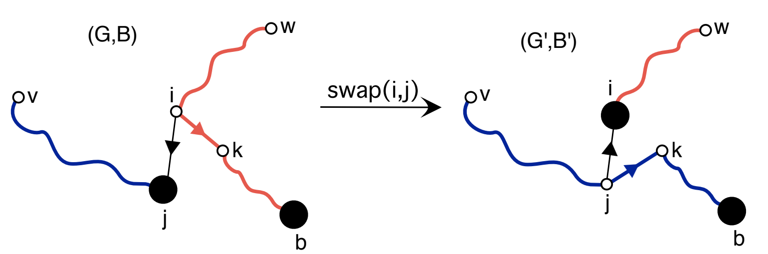

Figure 2: Case (iii): Rerouting v and w.

3 maximizing

In this section we study the question of when one can add edges to a planted graph without changing the corresponding cotransversal matroid. We prove that any planted graph (G, B) has a unique maximal planted graph $\overline{(G, B)}$ which contains it and has the same cotransversal matroid.

Definition 3.1. *Let $M = (E, \mathcal{B})$ be a matroid. Let $K \subseteq E$ and let B_K be a basis of K. The* contraction of M by K, *denoted M/K, is the matroid on $E - K$ whose bases are the sets $B' \subseteq E - K$ such that $B' \cup B_K$ is a basis of M.*

It is known [4] that any contraction $L(G, B)/K$ of a cotransversal matroid is also cotransversal. To obtain an explicit presentation of it, start with the planted graph (G, B). One can perform successive swaps until the resulting planted graph (G', B') is *complete with respect to* K, in the sense that G' contains no paths from K to $B' - K$; that is, B' contains $\text{rank}_{L(G,B)}(K)$ vertices from K. One then deletes from (G', B') the vertices in K and all the edges incident to them. It is easy to check that the resulting planted graph $(G', B') \backslash K$ is a presentation of the contraction $L(G, B)/K$.

Definition 3.2. *Let v be a vertex of a planted graph (G, B). The* claw *of v in (G, B) is $K_v = v \cup \{i \mid e_{vi} \in G\}$.*

Recall that a *loop* in a matroid is an element that does not occur in any basis of the matroid. A loop of the cotransversal matroid $L(G, B)$ is a vertex from which there is no path to B.

Proposition 3.3. *If (G, B) is a presentation of $L(G, B)$ then $(G, B) \cup e_{vw}$ is a presentation of $L(G, B)$ if and only if w is a loop in $L(G, B)/K_v$.*

Proof. We can perform swaps to make (G, B) into a complete presentation (G', B') with respect to K_v, without ever performing operations of the form $\mathbf{swap(v, b)}$. It follows that $(G, B) \cup e_{vw}$ and $(G', B') \cup e_{vw}$ can also be obtained from each other by a series of swaps. It then suffices to prove the statement in the case when (G, B) is complete with respect to K_v, and $(G, B) \backslash K_v$ is a presentation of $L(G, B)/K_v$.

If w is a loop in $L(G, B)/K_v$, we need to show that any set that can be routed to B in $(G, B) \cup e_{vw}$ can also be routed without using e_{vw}. Since $(G, B) \backslash K_v$ contains no paths from w to $B - K_v$, any path in (G, B) from w to some $b \in B$ must go through K_v. We then have that $b \in K_v$ since (G, B) is complete with respect to K_v. Consider some routing to B in the graph $(G, B) \cup e_{vw}$ where one of the paths ends at B and uses the edge e_{vw}. This path must end at some

$b \in K_v \cap B$, and can be shortened by going directly from v to b using the edge e_{vb}. This avoids the use of e_{vw} and does not introduce any intersection with the other paths of the routing.

Conversely, suppose there is a path P from w to $b \in B - K_v$ avoiding K_v. The set $(B \cap K_v) \cup v$ cannot be routed to B in (G, B) since (G, B) is complete with respect to K_v. However, this set can be routed to B in $(G, B) \cup e_{vw}$ by letting v follow the path $e_{vw} \cup P$. $\square$

Theorem 3.4. *For any planted graph (G, B) there exists a unique maximal planted graph $\overline{(G, B)}$ containing (G, B) such that $L(\overline{(G, B)}) = L(G, B)$.*

Proof. Given (G, B) we have a list E of edges we can add to (G, B) without changing the matroid. We claim $(G, B) \cup E = \overline{(G, B)}$. To see this, one checks that once we have added $F \subset E$ to (G, B) we can continue to add any edge in $E - F$ and we can not add any new ones. This follows from Proposition 3.3. $\square$

We are now ready to state our main theorem.

Theorem 3.5. *Two planted graphs (G, B) and (H, C) have the same cotransversal matroid if and only if $\overline{(H, C)}$ can be obtained from $\overline{(G, B)}$ by a series of swaps.*

The backward implication of Theorem 3.5 follows from Proposition 2.5. In order to prove the converse, we must first take a closer look at the duality between transversal and cotransversal matroids.

4 duality between cotransversal and transversal matroids

4.1 transversal matroids

Definition 4.1. *Let S be a finite set. Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a sequence of subsets of S. A* system of distinct representatives *or* SDR *of $\mathcal{A}$ is a sequence $(a_1, a_2, \ldots, a_r)$ where $a_i \neq a_j$ for $i \neq j$, and $a_i \in A_i$ for all i. A* transversal *is a set which can be ordered to obtain an SDR.*

Theorem 4.2. *[4] Given a sequence $\mathcal{A} = (A_1, \ldots, A_r)$ of subsets of a set S, there is a matroid on S whose bases are the transversals of $\mathcal{A}$. Any matroid that arises in this way is called a* transversal matroid*, and $\mathcal{A}$ is called a* presentation *of it.*

A presentation $\mathcal{A} = (A_1, \ldots, A_r)$ of a transversal matroid can also be described as a bipartite graph T between the "top" vertex set $[r] = \{1, \ldots, r\}$ and the "bottom" vertex set S, where top vertex i is connected to the elements of A_i for $1 \leq i \leq r$. The SDRs of $\mathcal{A}$ become perfect matchings of $[r]$ into S in the graph T. We will use these two points of view interchangeably.

Theorem 4.3. *[2] For every family $\mathcal{A} = (A_1, \ldots, A_n)$ of subsets of a set S there is a unique maximal family $\overline{\mathcal{A}} = (\overline{A_1}, \ldots, \overline{A_n})$ of subsets of S such that $A_i \subseteq \overline{A_i}$ for $1 \le i \le n$, and $\mathcal{A}$ and $\overline{\mathcal{A}}$ give rise to the same transversal matroid.*

The following lemma on SDRs will be crucial later on.

Lemma 4.4 (SDR exchange lemma)**.** *Suppose that a bipartite graph, corresponding to the sequence $\mathcal{A} = (A_1, \ldots, A_r)$ of sets, satisfies the* dragon marriage condition:[1] *for all $\{i_1, \ldots, i_k\} \subseteq [r]$ we have $|A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}| \ge k+1$. Then for any two SDRs M and M' of $\mathcal{A}$, there is a sequence $M = M_1, \ldots, M_s = M'$ of SDRs of $\mathcal{A}$ such that M_i and M_{i+1} differ in exactly one position for $1 \le i \le s-1$.*

Proof. Construct a graph H in which the vertices are the SDRs of $\mathcal{A}$ and two SDRs are connected by an edge if they differ in only one position. We need to prove that H is connected.

Suppose H is not connected. Consider two SDRs $M_b = (b_1, \ldots, b_r) \in H_1$ and $M_c = (c_1, \ldots, c_r) \in H_2$ which are in distinct components of H, chosen in such a way that $|M_b - M_c|$ is minimal. We consider the following two cases.

(i) If $\{b_1, \ldots, b_r\} \ne \{c_1, \ldots, c_r\}$, then for some i we have $b_i \notin \{c_1, \ldots, c_r\}$. Then $M_c' = (c_1, \ldots, b_i, \ldots, c_r)$ is an SDR in the connected component of M_c, and satisfies $|M_b - M_c'| < |M_b - M_c|$.

(ii) Suppose $\{b_1, \ldots, b_r\} = \{c_1, \ldots, c_r\}$. We can partition the vertices of our bipartite graph T into three parts based on the matchings M_b and M_c, which are shown in Figure 3. (The dotted edges will be explained later.) Part I consists of the vertices of T that are neither in M_b nor in M_c. Part II consists of the top vertices i such that $b_i = c_i$, and the bottom vertices matched to them. Part III consists of the remaining vertices.

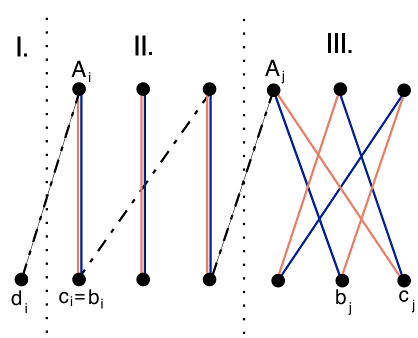

Figure 3: Case (ii): T is partitioned into three pieces according to M_b and M_c.

[1] This name is due to Postnikov; see [5] for the reasoning behind it.

The dragon marriage condition gives $|S| \geq r+1$, so there is some $d_i \in A_i$ such that $d_i \notin \{b_1, \ldots, b_r\}$. Therefore $M_b' = (b_1, \ldots, d_i, \ldots, b_r)$ and $M_c' = (c_1, \ldots, d_i, \ldots, c_r)$ are SDRs which are in the connected components of M_b and M_c. We must have $b_i = c_i$, or else $|M_b' - M_c'| < |M_b - M_c|$. In Figure 3, this means that there are no edges from the top of Part III to Part I.

By the dragon marriage condition, the top of Part III must be connected to the bottom of Part II. Define a *zigzag path* to be a path such that:

- its starting point is a vertex in the top of Part III,
- this is the only vertex of Part III it contains, and
- every second edge is a common edge of the matchings M_b and M_c.

We claim that there is at least one zigzag path that ends in Part I. To verify this, consider the set V of vertices in the top that can be reached by a zigzag path starting from the top of Part III. Notice that every top vertex in Part III is in V. By the dragon marriage condition, some vertex in V must be connected to a vertex d in the bottom of the graph that is not matched to V in M_b and M_c. If d was in Part II, it would be matched in M_b and M_c to a top vertex $A \notin V$; the edge from d to A would complete a zigzag path that contains A, contradicting our definition of the set V. Therefore d is in Part I.

Consider a zigzag path to d starting at A_j, as shown in Figure 3. Now construct new SDRs M_b' and M_c' by unlinking b_j and c_j from A_j in M_b and M_c respectively, as well as all the edges of M_b and M_c along the zigzag path P. Instead, in both M_b and M_c, rematch the vertices along the edges of path P which were not used by M_b and M_c; these are dotted in Figure 3. Figure 4 shows the resulting new matchings M_b' and M_c' in this example. Now notice that

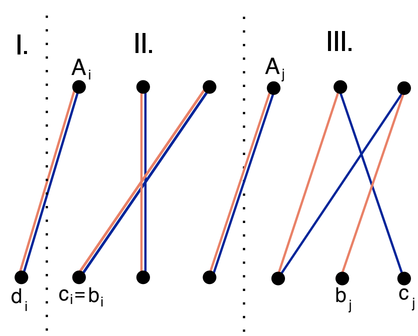

Figure 4: The new matchings M_b' and M_c'.

$|M_b' - M_c'| < |M_b - M_c|$, and M_b' and M_c' are in the same connected components of H as M_b and M_c, respectively. This is a contradiction, and we conclude that H is connected. $\square$

4.2 duality

Definition 4.5. *If $M = (E, \mathcal{B})$ is a matroid then $\mathcal{B}^* = \{E - B \mid B \in \mathcal{B}\}$ is also the collection of bases for a matroid $M^* = (E, \mathcal{B}^*)$, called the* dual *of M.*

Theorem 4.6. *(Ingleton, Piff) [1, 3, 4] Cotransversal matroids are precisely the duals of transversal matroids.*

To prove our main result, Theorem 3.5, we need to take a closer look at how a presentation of a transversal matroid is determined by a presentation of its dual cotransversal matroid.

Given a planted graph presentation (G, B) with vertex set V of a cotransversal matroid $L(G, B)$, we can construct a presentation $\mathcal{A}$ of its dual transversal matroid $L(G, B)^*$, by the following method. For each $i \in V - B$, define $A_i := \{i\} \cup \{u \mid e_{iu} \in L(G, B)\}$. The sets A_i with $i \in V - B$ make up a presentation of $L^*(G, B)$.

In the opposite direction, each SDR of a transversal matroid leads to a presentation of the dual cotransversal matroid as follows. Consider a presentation $\mathcal{A} = (A_1, A_2, \ldots, A_k)$ of a transversal matroid; say $\bigcup_{i=1}^{r} A_i = [n]$. If $(a_1, \ldots, a_k)$ is an SDR of $\mathcal{A}$, construct a planted graph (G, B) as follows. Let $[n]$ be the set of vertices of G, and whenever $x \in A_j$ and $x \neq a_j$, draw the directed edge from a_j to x. Let B be the complement of $\{a_1, \ldots, a_k\}$. This will give a presentation of the dual cotransversal matroid.

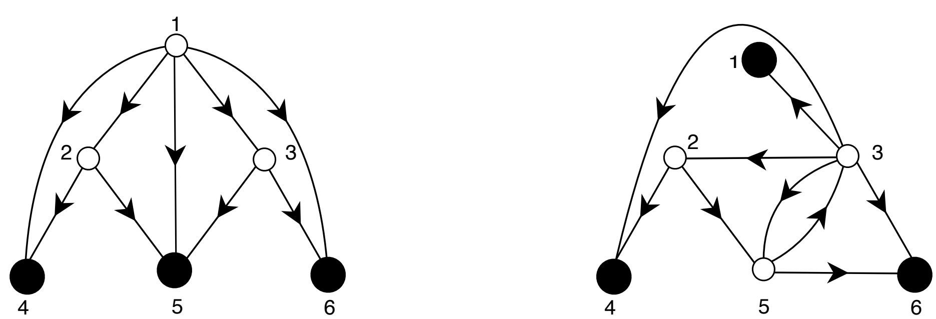

Figure 5: The planted graphs given by $\mathcal{A} = (\{1, 2, 3, 4, 5, 6\}, \{2, 4, 5\}, \{3, 5, 6\})$ with SDRs $(1, 2, 3)$ and $(3, 2, 5)$, respectively. They are presentations of the cotransversal matroid dual to the matroid of $\mathcal{A}$.

Figure 5 shows the planted graphs that arise from two SDRs of the same collection of sets.

5 the proof

We have now done all the necessary work to prove our main result.

Proof of Theorem 3.5. As we already mentioned, the backward direction of Theorem 3.5 follows from Proposition 2.5. Now suppose (G, B) and (H, C) are maximal presentations of the same cotransversal matroid M. When we apply the procedure described in Section 4.2 to them, both of them must give rise to the unique maximal presentation $\mathcal{A}$ of the dual transversal matroid M^*. They correspond to different matchings M_1 and M_2 of $\mathcal{A}$.

Since $\mathcal{A}$ has at least one matching, we have $|A_{i_1} \cup \cdots \cup A_{i_k}| \geq k$ for all $\{i_1, \ldots, i_k\}$. If we have $|A_{i_1} \cup \cdots \cup A_{i_k}| = k$ for some $\{i_1, \ldots, i_k\}$, then all the elements of $A_{i_1} \cup \cdots \cup A_{i_k}$ are in every basis of M^*, and are loops in M. By maximality, the loops of M form complete subgraphs in both (G, B) and (H, C). We can then restrict our attention to the non-loops of M, where the dragon marriage condition is satisfied.

Applying Lemma 4.4, we can get from M_1 to M_2 by exchanging one element of the matching at a time. These matching exchanges in the bipartite graph correspond exactly to swaps in the corresponding planted graphs. It follows that one can get from (G, B) to (H, C) by a series of swaps, as desired. $\square$

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