

# THE DOUBLE GROMOV-WITTEN INVARIANTS OF HIRZEBRUCH SURFACES ARE PIECEWISE POLYNOMIAL

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ABSTRACT. We define the *double Gromov-Witten invariants* of Hirzebruch surfaces in analogy with double Hurwitz numbers, and we prove that they satisfy a piecewise polynomiality property analogous to their 1-dimensional counterpart. Furthermore we show that each polynomial piece is either even or odd, and we compute its degree. Our methods combine floor diagrams and Ehrhart theory.

## 1. INTRODUCTION

*Hurwitz numbers* count the holomorphic maps  $C \rightarrow \mathbb{C}P^1$  of a fixed degree  $d$ , with prescribed ramification values, and prescribed ramification profiles over each ramification value. These numbers are connected to several areas of mathematics including algebraic geometry, combinatorics, and representation theory, among others. In particular, the ELSV formula [ELSV01] relates *simple* Hurwitz numbers (where there is a single critical value with possibly less than  $d - 1$  preimages) to the moduli spaces of complex algebraic curves  $\overline{\mathcal{M}}_{g,n}$ .

No generalization of the ELSV-formula is known yet for *double* Hurwitz numbers; however, as possible evidence toward such a generalization, these numbers enjoy a very rich structure. In particular, Goulden, Jackson, and Vakil proved, among other things, that double Hurwitz numbers are piecewise polynomial [GJV05]. Later on, Cavalieri, Johnson, and Markwig used tropical geometry to give a new proof of this piecewise polynomiality [CJM10, CJM11]. In addition, they found a wall crossing formula giving the difference of double Hurwitz numbers between two adjacent chambers of polynomiality, generalizing the formula in genus 0 proved in [SSV08]. An alternative approach to prove piecewise polynomiality and obtain these wall crossing formulas, based on the De Concini-Procesi-Vergne theory of *remarkable spaces* [DCPV10a, DCPV10b], was proposed by the first author in [Ard09].

*Results.* In this note we introduce the *double Gromov-Witten invariants of Hirzebruch surfaces*, which generalize double Hurwitz numbers, and we establish their piecewise polynomiality. We now make these assertions more precise, referring to Section 2 for precise definitions.

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Given  $k \geq 0$ , we denote by  $\mathbb{F}_k = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(k) \oplus \mathcal{O}_{\mathbb{C}P^1})$  the  $k$ th Hirzebruch surface. We say that an algebraic curve in  $\mathbb{F}_k$  is of bidegree  $(a, b)$  if it is linearly equivalent to the union of  $a$  copies of the section  $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(k) \oplus \{0\})$  together with  $b$  copies of a fiber (see Section 2.1). The double Gromov-Witten invariants of  $\mathbb{F}_k$ , denoted by  $N_g^{\alpha, \beta, \tilde{\alpha}, \tilde{\beta}}(a, b, k)$ , count algebraic curves in  $\mathbb{F}_k$  of a given bidegree  $(a, b)$  and genus  $g$ , passing through an appropriate configuration of points, and having fixed intersection patterns  $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$  with the sections  $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(k) \oplus \{0\})$  and  $\mathbb{P}(\{0\} \oplus \mathcal{O}_{\mathbb{C}P^1})$ .

We encode the double Gromov-Witten invariants of the Hirzebruch surface  $\mathbb{F}_k$  in a function  $F_{a,k,g}^{n_1, n_2}(\mathbf{x}, \mathbf{y})$  as follows. Let us fix  $a > 0$  and  $k, g \geq 0$  as above, and let us also fix two additional non-negative integer numbers  $n_1$  and  $n_2$ . We then define

$$\Lambda = \left\{ (x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \mid \sum x_i + \sum y_j + ak = 0 \right\} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

Given an element  $(\mathbf{x}, \mathbf{y}) = ((x_1, \dots, x_{n_1}), (y_1, \dots, y_{n_2})) \in \Lambda$ , we store the multiplicities of the entries of  $(\mathbf{x}, \mathbf{y})$  in the four sequences  $\alpha = (\alpha_i)_{i \geq 1}, \beta = (\beta_i)_{i \geq 1}, \tilde{\alpha} = (\tilde{\alpha}_i)_{i \geq 1}$ , and  $\tilde{\beta} = (\tilde{\beta}_i)_{i \geq 1}$  of non-negative integers, where  $\alpha_i$  is the number of elements  $x_j$  equal to  $i$ ,  $\beta_i$  is the number of elements  $y_j$  equal to  $i$ ,  $\tilde{\alpha}_i$  is the number of elements  $x_j$  equal to  $-i$ , and  $\tilde{\beta}_i$  is the number of elements  $y_j$  equal to  $-i$ . Set  $b = \sum(\tilde{\alpha}_i + \tilde{\beta}_i)$ . In our examples, we will omit the parentheses in  $\alpha, \tilde{\alpha}, \beta$ , and  $\tilde{\beta}$  to simplify the notation. For instance, the vector  $\alpha = (2, 0, 1)$  will be denoted by  $\alpha = 201$ .

**Definition 1.1.** The function  $F_{a,k,g}^{n_1, n_2}(\mathbf{x}, \mathbf{y})$  is defined by:

$$F_{a,k,g}^{n_1, n_2} : \begin{array}{ccc} \Lambda & \longrightarrow & \mathbb{Z} \\ (\mathbf{x}, \mathbf{y}) & \longmapsto & N_g^{\alpha, \beta, \tilde{\alpha}, \tilde{\beta}}(a, b, k) \end{array} .$$

**Example 1.2.** We have

$$F_{3,2,1}^{4,5}((-2, -2, -1, 1), (-3, -1, -1, 1, 2)) = N_1^{12, 201, 1, 11}(3, 4, 2)$$

because the multiplicities in  $(\mathbf{x}, \mathbf{y}) = ((-2, -2, -1, 1), (-3, -1, -1, 1, 2))$  are given by  $(\alpha, \beta, \tilde{\alpha}, \tilde{\beta}) = (12, 201, 1, 11)$ . Since the superscript of  $F_{3,2,1}$  denotes the sizes of the input vectors, we can drop it and write  $F_{3,2,1}((-2, -2, -1, 1), (-3, -1, -1, 1, 2))$ .

The following theorems are the main results of this note.

**Theorem 1.3.** Let  $k, g, n_1, n_2 \geq 0$  and  $a \geq 1$  be fixed integers. The function

$$F_{a,k,g}^{n_1, n_2}(\mathbf{x}, \mathbf{y})$$

of double Gromov-Witten invariants of the Hirzebruch surface  $\mathbb{F}_k$  is piecewise polynomial in the chambers of the hyperplane arrangement

$$\begin{aligned} \sum_{i \in S} x_i + \sum_{j \in T} y_j + kr = 0 & \quad (S \subseteq [n_1], T \subseteq [n_2], 0 \leq r \leq a), \\ y_i - y_j = 0 & \quad (1 \leq i < j \leq n_2) \end{aligned}$$

inside  $\Lambda = \{(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \mid \sum x_i + \sum y_i + ak = 0\} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ .

**Theorem 1.4.** Each polynomial piece of  $F_{a,k,g}^{n_1, n_2}(\mathbf{x}, \mathbf{y})$  has degree  $n_2 + 3g + 2a - 2$ , and is either even or odd.

Theorems 1.3 and 1.4 might suggest that a 2-dimensional generalization of the ELSV-formula could exist.

*Techniques.* Our proofs of Theorems 1.3 and 1.4 combine the enumeration of complex curves in complex surfaces via *floor diagrams*, together with the approach proposed in [Ard09, CJM11] to study the piecewise polynomiality of double Hurwitz numbers. Floor diagrams were introduced in [BM08, BM07, BM], and further explored in [AB13, ABLdM11, Blo11, BCK13, BGM12, BG14b, BG14a, BP13, BBM14, Bru14, FM10, Liu13, LO14]. They allow one to replace the geometric enumeration of curves by a purely combinatorial problem, applying the following general strategy: suppose that one wants to enumerate algebraic curves in some complex surface  $X$  interpolating a configuration of points  $\mathcal{P}$ ; choose a non-singular rational curve  $E$  in  $X$ , and degenerate  $X$  into the union of  $X$  together with a chain of copies of the compactified normal bundle  $\mathcal{N}_E$  of  $E$  in  $X$ ; moreover, specialize exactly one point of  $\mathcal{P}$  to each of these copies of  $\mathcal{N}_E$ ; now floor diagrams encode the limit of curves under enumeration in this degeneration process. In good situations, including the one we deal with here, all limit curves can be completely recovered only from the combinatorics of the floor diagrams. We refer to [Bru14, Section 1.1] for more details about the heuristic of the floor decomposition technique.

It turns out that the tropical count of double Hurwitz numbers performed in [CJM10] can be interpreted as a floor diagram count in dimension 1. The underlying combinatorial objects are very similar, and thus we are able to transpose part of the approach from [Ard09, CJM11] to the 2-dimensional case. The key idea is to interpret the relevant combinatorial problem as the (weighted) enumeration of lattice points in flow polytopes, and apply techniques from Ehrhart theory.

*Organization.* The paper is organized as follows. Double Gromov-Witten invariants of Hirzebruch surfaces are defined in Section 2, and we explain how to compute them via floor diagrams in Section 3. This reduces our enumerative geometric question to a combinatorial question, which we treat in the remaining sections. Section 4 recalls and extends some facts from Ehrhart theory. We use these results in Section 5, where we rephrase the enumeration of floor diagrams in terms of polyhedral geometry, and complete the proof of our main results. We work out a concrete example in Section 6, and end the paper in Section 7 with some concluding remarks about possible extensions of this work.

## 2. DOUBLE GROMOV-WITTEN INVARIANTS OF HIRZEBRUCH SURFACES

In this section, to state our results, we will require some familiarity with the geometry of complex curves; for an introduction, see [GH94, Bea83]. For more details about the enumerative geometry of Hirzebruch surfaces, we refer to [Vak00].

**2.1. Hirzebruch surfaces.** Recall the  $k$ th Hirzebruch surface, with  $k \geq 0$ , is denoted by  $\mathbb{F}_k$ , i.e.  $\mathbb{F}_k = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(k) \oplus \mathcal{O}_{\mathbb{C}P^1})$ . Any compact complex surface admitting a holomorphic fibration to  $\mathbb{C}P^1$  with fiber  $\mathbb{C}P^1$  is isomorphic to exactly one of the Hirzebruch surfaces.

For example one has  $\mathbb{F}_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ . The surface  $\mathbb{F}_1$  is the projective plane blown up at a point, and  $\mathbb{F}_2$  is the quadratic cone with equation  $x^2 + y^2 + z^2$  in  $\mathbb{C}P^3$  blown up at the node. In the last two cases the fibration is given by the extension of the projection from the blown-up point to a line (if  $k = 1$ ) or a hyperplane section (if  $k = 2$ ) which does not pass through the blown-up point.

Let us denote by  $B_k$  (resp.  $E_k$ , and  $F_k$ ) the section  $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(k) \oplus \{0\})$  (resp. the section  $\mathbb{P}(\{0\} \oplus \mathcal{O}_{\mathbb{C}P^1})$ , and a fiber). The curves  $B_k$ ,  $E_k$ , and  $F_k$  have self-intersections  $B_k^2 = k$ ,  $E_k^2 = -k$ , and  $F_k^2 = 0$ . When  $k \geq 1$ , the curve  $E_k$  itself determines uniquely the Hirzebruch surface, since it is the only reduced and irreducible algebraic curve in  $\mathbb{F}_k$  with negative self-intersection. The group  $\text{Pic}(\mathbb{F}_k) = H_2(\mathbb{F}_k, \mathbb{Z})$  is isomorphic to  $\mathbb{Z} \times \mathbb{Z}$  and is generated by the classes of  $B_k$  and  $F_k$ . Note that we have  $E_k = B_k - kF_k$  in  $H_2(\mathbb{F}_k, \mathbb{Z})$ . An algebraic curve  $C$  in  $\mathbb{F}_k$  is said to be of *bidegree*  $(a, b)$  if it realizes the homology class  $aB_k + bF_k$  in  $H_2(\mathbb{F}_k, \mathbb{Z})$ . By the adjunction formula, a non-singular algebraic curve  $C$  of bidegree  $(a, b)$  in  $\mathbb{F}_k$  has genus

$$g(C) = \frac{a(a-1)}{2}k + ab - a - b + 1.$$

**2.2. Double Gromov-Witten invariants.** Let us fix four integers<sup>1</sup>  $a > 0$ , and  $b, k, g \geq 0$ , as well as four sequences of non-negative integers  $\alpha = (\alpha_i)_{i \geq 1}$ ,  $\tilde{\alpha} = (\tilde{\alpha}_i)_{i \geq 1}$ ,  $\beta = (\beta_i)_{i \geq 1}$ ,  $\tilde{\beta} = (\tilde{\beta}_i)_{i \geq 1}$  such that

$$\sum_i (\alpha_i + \beta_i) = ak + b \quad \text{and} \quad \sum_i (\tilde{\alpha}_i + \tilde{\beta}_i) = b.$$

In particular this implies that only finitely many terms of the four sequences are non-zero. We define  $l = 2a + g + \sum (\beta_i + \tilde{\beta}_i) - 1$ .

Next, let us choose a generic configuration

$$\omega = \{q_1^1, \dots, q_{\alpha_1}^1, \dots, q_1^i, \dots, q_{\alpha_i}^i, \dots, p_1, \dots, p_l, \tilde{q}_1^1, \dots, \tilde{q}_{\tilde{\alpha}_1}^1, \dots, \tilde{q}_1^i, \dots, \tilde{q}_{\tilde{\alpha}_i}^i, \dots\}$$

of  $l + \sum (\alpha_i + \tilde{\alpha}_i)$  points in  $\mathbb{F}_k$  such that  $q_j^i \in B_k$ ,  $\tilde{q}_j^i \in E_k$ , and  $p_i \in \mathbb{F}_k \setminus (B_k \cup E_k)$ .

We denote by  $N_g^{\alpha, \beta, \tilde{\alpha}, \tilde{\beta}}(a, b, k)$  the number of irreducible complex algebraic curves  $C$  in  $\mathbb{F}_k$  of genus  $g$  such that

- (1)  $C$  passes through all the points  $q_j^i$ ,  $\tilde{q}_j^i$ , and  $p_i$ ;
- (2)  $C$  has order of contact  $i$  with  $B_k$  at  $q_j^i$ , and has  $\beta_i$  other (non-prescribed) points with order of contact  $i$  with  $B_k$ ;
- (3)  $C$  has order of contact  $i$  with  $E_k$  at  $\tilde{q}_j^i$ , and has  $\tilde{\beta}_i$  other (non-prescribed) points with order of contact  $i$  with  $E_k$ .

This number is finite and doesn't depend on the chosen generic configuration of points. We call this number a *double Gromov-Witten invariant* of  $\mathbb{F}_k$  in analogy with double Hurwitz numbers.

<sup>1</sup>As we will see in Remark 2.2, one can extend our definition of Gromov-Witten invariants to the case  $a = 0$  with some extra care.

**Example 2.1.** Recall that we omit the parentheses in  $\alpha, \tilde{\alpha}, \beta$ , and  $\tilde{\beta}$  to simplify the notation. By the adjunction formula, one has  $N_g^{\alpha, \beta, \tilde{\alpha}, \tilde{\beta}}(a, b, k) = 0$  as soon as  $g > \frac{a(a-1)}{2}k + ab - a - b + 1$ .

It is well known that there exist exactly 2 conics in  $\mathbb{C}P^2$  passing through 4 points in generic positions and tangent to a fixed line. In our notation, this translates to  $N_0^{0,0,0,0}(2, 0, 1) = 2$ .

If we now look at conics passing through 3 points in generic position in  $\mathbb{C}P^2$  and tangent to a fixed line at a prescribed point, then we find exactly one such conic, i.e.  $N_0^{0,0,0,0}(2, 0, 1) = 1$ .

As less straightforward examples, we give the values  $N_0^{0,0,0,0}(2, 2, 0) = 8$  and  $N_0^{0,1,0,1}(3, 1, 1) = 8$ . We will compute these numbers in the next section, using floor diagrams.

**Remark 2.2.** Theorem 1.3 extends trivially to the case  $a = 0$ . However, for simplicity we chose to define the invariants  $N_g^{\alpha, \beta, \tilde{\alpha}, \tilde{\beta}}(a, b, k)$  by counting immersed algebraic curves instead of maps. As a consequence, the cases  $a = 0$  and  $b > 1$  are problematic with our simplified definition because of the appearance of non-reduced curves. However with the suitable definition of Gromov-Witten invariants in terms of maps, we have (see [Vak00, Section 8])

$$N_0^{u_b, 0, 0, u_b}(0, b, k) = N_0^{0, u_b, u_b, 0}(0, b, k) = \frac{1}{b} \quad \text{and} \quad N_0^{0, u_b, 0, u_b}(0, b, k) = 1,$$

where  $u_b$  is the  $b$ -th vector of the canonical basis of  $\mathbb{R}^n$ , and

$$N_g^{\alpha, \beta, \tilde{\alpha}, \tilde{\beta}}(0, b, k) = 0$$

in all the other cases. In particular, the conclusion of Theorem 1.3 holds when  $a = 0$  and either  $g \neq 0$  or  $n_1 = 0$ . Note that in the case when  $a = g = 0$  and  $n_1 = 1$ , the function

$$F_{0, k, 0}^{1, 1}(\pm b, \mp b) = \frac{1}{b}$$

is rational instead of polynomial.

### 3. FLOOR DIAGRAMS

Here we recall how to enumerate complex curves in  $\mathbb{F}_k$  using floor diagrams. We use notation inspired by [FM10, AB13].

**Definition 3.1.** A *marked floor diagram*  $\mathcal{D}$  for  $\mathbb{F}_k$  consists of

- (1) A vertex set  $V = L \sqcup C \sqcup R$  where  $C$  is totally ordered from left to right,  $L = \{q_1, \dots, q_l\}$  is unordered and to the left of  $C$ , and  $R = \{\tilde{q}_1, \dots, \tilde{q}_r\}$  is unordered and to the right of  $C$ .
- (2) A coloring of the vertices with black, white, and gray, so that every vertex in  $L$  and  $R$  is white.
- (3) A set  $E$  of edges, directed from left to right, such that
  - The resulting graph is connected.
  - Every white vertex has exactly one edge, which connects it to a black vertex.

- Every gray vertex has exactly two edges; one coming from a black vertex, and the other one going to a black vertex.
- (4) A choice of positive integer weights  $w(e)$  on each edge  $e$  such that if we define the *divergence* of a vertex  $v$  to be

$$\operatorname{div}(v) := \sum_{\substack{\text{edges } e \\ v' \xrightarrow{e} v}} w(e) - \sum_{\substack{\text{edges } e \\ v \xrightarrow{e} v'}} w(e).$$

then

- $\operatorname{div}(v) = k$  for every black vertex  $v$ .
- $\operatorname{div}(v) = 0$  for every gray vertex  $v$ .

**Example 3.2.** Figure 1 shows a floor diagram for  $\mathbb{F}_2$ . We draw dotted lines to separate  $L$ ,  $C$ , and  $R$ . To simplify our pictures, we omit the labels of the vertices in  $L$  and  $R$ . For instance, the picture of Figure 1 actually represents three different floor diagrams, corresponding to the three different ways of assigning the labels  $q_1, q_2$ , and  $q_3$  to the vertices in  $L$ . When an edge  $e$  has weight  $w(e) > 1$ , we write that weight next to it. Since there is no risk of confusion, we omit the (left-to-right) edge directions. Notice that  $\operatorname{div}(v) = 2$  for every black vertex and  $\operatorname{div}(v) = 0$  for all the gray vertices.

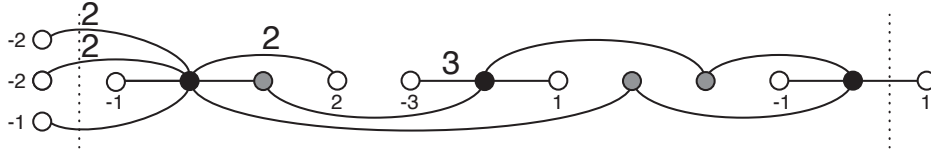


FIGURE 1. A floor diagram for the Hirzebruch surface  $\mathbb{F}_2$ .

We associate several parameters to a marked floor diagram  $\mathcal{D}$ :

- The *type* of  $\mathcal{D}$  is the pair  $(n_1, n_2)$  where  $\mathcal{D}$  has  $n_1$  white vertices in  $L \cup R$  (i.e.  $n_1 = l + r$ ), and  $n_2$  white vertices in  $C$ .
  - The *divergence sequence* is a vector  $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2}$  of length  $n_1 + n_2$  where
    - $\mathbf{x} = (\operatorname{div}(q_1), \dots, \operatorname{div}(q_l), \operatorname{div}(\tilde{q}_1), \dots, \operatorname{div}(\tilde{q}_r))$  is the sequence of divergences of vertices in  $L$  and  $R$ ;
    - $\mathbf{y}$  is the sequence of divergences of white vertices in  $C$ , listed from left to right.
- Since the sum of all the divergences in the graph must be 0, we must have

$$\sum x_i + \sum y_j = -ka$$

where  $a$  is the number of black vertices of  $\mathcal{D}$ . The vector  $\mathbf{x}$  is called the *left-right sequence* of  $\mathcal{D}$ .

- The *divergence multiplicity vector* is a sequence of four vectors  $(\alpha(\mathbf{x}), \beta(\mathbf{y}), \tilde{\alpha}(\mathbf{x}), \tilde{\beta}(\mathbf{y}))$  where

- $\alpha_i$  is the number of (white) vertices  $v$  in  $L$  with  $\operatorname{div}(v) = -i$ ;
- $\tilde{\alpha}_i$  is the number of (white) vertices  $v$  in  $R$  with  $\operatorname{div}(v) = i$ ;

- $\beta_i$  is the number of white vertices  $v$  in  $C$  with  $\text{div}(v) = -i$ ;
- $\tilde{\beta}_i$  is the number of white vertices  $w$  in  $C$  with  $\text{div}(v) = i$ .

Clearly  $(\mathbf{x}, \mathbf{y})$  determines  $(\alpha, \beta, \tilde{\alpha}, \tilde{\beta})$  completely, while  $(\alpha, \beta, \tilde{\alpha}, \tilde{\beta})$  determines  $(\mathbf{x}, \mathbf{y})$  up to the order of the coordinates in  $\mathbf{x}$  and  $\mathbf{y}$ .

◦ The *bidegree* of  $\mathcal{D}$  is the pair  $(a, b)$  of positive integers such that

$$\sum_i i(\tilde{\alpha}_i + \tilde{\beta}_i) = b, \quad \sum_i i(\alpha_i + \beta_i) = ka + b.$$

Recall that  $a$  is the number of black vertices of  $\mathcal{D}$ .

- The *genus*  $g(\mathcal{D})$  of  $\mathcal{D}$  is its first Betti number; it equals  $g(\mathcal{D}) = 1 - |V| + |E|$ .
- The *multiplicity*  $\mu(\mathcal{D})$  is the product of the internal edge weights, where an edge is *internal* if it connects two vertices of  $C$ .

**Example 3.3.** Suppose that the vertices of the floor diagram for  $\mathbb{F}_2$  in Figure 1 are, from the top to the bottom,  $q_1, q_2$ , and  $q_3$ . Then its divergence sequence is

$$((-2, -2, -1, 1), (-1, 2, -3, 1, -1)).$$

The divergences of the white vertices in  $L$  and  $R$  are  $-2, -2, -1$  and  $1$ ; they are encoded, respectively, in the vectors  $\alpha = 12, \tilde{\alpha} = 1$ . The divergences of the white vertices in  $C$  are  $-1, -1, -3$ , and  $1, 2$ ; they are respectively encoded in the vectors  $\beta = 201, \tilde{\beta} = 11$ . Therefore the divergence multiplicity vector is

$$(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}) = (12, 1, 201, 11).$$

The negative white divergences add up to  $-\sum_i i(\alpha_i + \beta_i) = -10 = -(2a + b)$  and the positive white divergences add up to  $\sum_i i(\tilde{\alpha}_i + \tilde{\beta}_i) = 4 = b$ . Therefore  $\mathcal{D}$  has bidegree  $(a, b) = (3, 4)$ . Visibly, the genus is  $g(\mathcal{D}) = 1$  and the multiplicity is  $\mu(\mathcal{D}) = 6$ .

In Definition 3.1, we use the adjective *marked* in reference to the corresponding objects in [BM07, FM10, AB13], which are *floor diagrams* endowed with additional structure. However, since we do not consider unmarked floor diagram in this note, we will abbreviate *marked floor diagram* to *floor diagram* in the rest of the text.

The following theorem is a very minor variation on [BM08, Theorem 3.6]. It replaces the enumeration of algebraic curves by the enumeration of floor diagrams.

**Theorem 3.4.** *Let  $a > 0$  and  $b, k, g \geq 0$  be four integer numbers, and  $\mathbf{x}$  a vector with coordinates in  $\mathbb{Z} \setminus \{0\}$ . We write  $\alpha(\mathbf{x}) = \alpha$  and  $\tilde{\alpha}(\mathbf{x}) = \tilde{\alpha}$ . Then for any two sequences of non-negative integer numbers  $\beta = (\beta_i)_{i \geq 1}$  and  $\tilde{\beta} = (\tilde{\beta}_i)_{i \geq 1}$  such that*

$$\sum_i i(\alpha_i + \beta_i) = ak + b, \quad \text{and} \quad \sum_i i(\tilde{\alpha}_i + \tilde{\beta}_i) = b,$$

one has

$$N_g^{\alpha, \beta, \tilde{\alpha}, \tilde{\beta}}(a, b, k) = \sum_{\mathcal{D}} \mu(\mathcal{D}),$$

where the sum runs over all floor diagrams  $\mathcal{D}$  of bidegree  $(a, b)$ , genus  $g$ , left-right sequence  $\mathbf{x}$ , and divergence multiplicity vector  $(\alpha, \beta, \tilde{\alpha}, \tilde{\beta})$  for  $\mathbb{F}_k$ .

*Proof.* Strictly speaking, the tropical proof of [BM08, Theorem 3.6] uses Mikhalkin's Correspondence Theorem [Mik05, Theorem 1], and proves our theorem only when

$$\alpha = \tilde{\alpha} = 0, \quad \beta = (ka + b, 0, \dots, 0), \quad \text{and} \quad \tilde{\beta} = (b, 0, \dots, 0).$$

A generalization of Mikhalkin's Correspondence Theorem which covers the case of curves satisfying tangency conditions with toric divisors can be found for example in [Shu12, Theorem 2]. In particular, the generalization of the proof of [BM08, Theorem 3.6] to our situation is straightforward. Alternatively, a proof of Theorem 3.4 within classical algebraic geometry can be obtained by a straightforward adaptation of [Bru14, Section 5.2].  $\square$

**Example 3.5.** Figure 2a represents the only floor diagram in  $\mathbb{F}_1$  of bidegree  $(2, 0)$ , genus 0, and divergence multiplicity vector  $(0, 01, 0, 0)$ . Hence  $N_0^{0,01,0,0}(2, 0, 1) = 2$  according to Theorem 3.4.

**Example 3.6.** Figure 2b represents the only floor diagram in  $\mathbb{F}_1$  of bidegree  $(2, 0)$ , genus 0, and divergence multiplicity vector  $(01, 0, 0, 0)$ . By Theorem 3.4, we have  $N_0^{01,0,0,0}(2, 0, 1) = 1$ .

**Example 3.7.** Figure 2c represents the only floor diagram in  $\mathbb{F}_0$  of bidegree  $(2, 2)$ , genus 0, and divergence multiplicity vector  $(01, 0, 0, 01)$ . Therefore by Theorem 3.4, we have  $N_0^{01,0,0,01}(2, 2, 0) = 8$ .

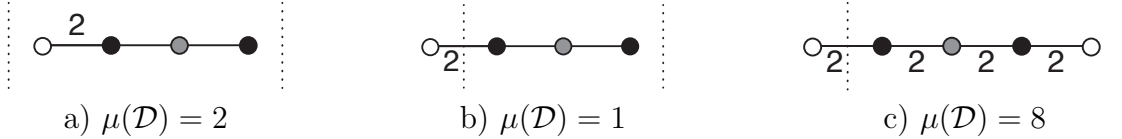


FIGURE 2. Computation of  $N_0^{0,01,0,0}(2, 0, 1) = 2$ ,  $N_0^{01,0,0,0}(2, 0, 1) = 1$  and  $N_0^{01,0,0,01}(2, 2, 0) = 8$ .

**Example 3.8.** Figure 3 represents all floor diagrams in  $\mathbb{F}_1$  of bidegree  $(3, 1)$ , genus 0, and divergence multiplicity vector  $(01, 1, 1, 0)$ . By Theorem 3.4, we have

$$N_0^{(01,1,1,0)}(3, 1, 1) = 1 + 1 + 1 + 4 + 1 = 8.$$

#### 4. WEIGHTED PARTITION FUNCTIONS AND WEIGHTED EHRHART RECIPROCITY

In this section we collect a few results about partition functions and their weighted counterparts that will be useful to us in what follows.

Let  $X = \{a_1, \dots, a_m\} \subset \mathbb{Z}^d$  be a finite multiset of lattice vectors in  $\mathbb{R}^d$ . The *rank*  $r(X)$  of  $X$  is the dimension of the real span of  $X$ . We may regard  $X$  as an  $m \times d$  matrix whose columns are  $a_1, \dots, a_m$ . We say  $X$  is *unimodular* if all the maximal minors are equal to  $-1, 0$ , or  $1$ .



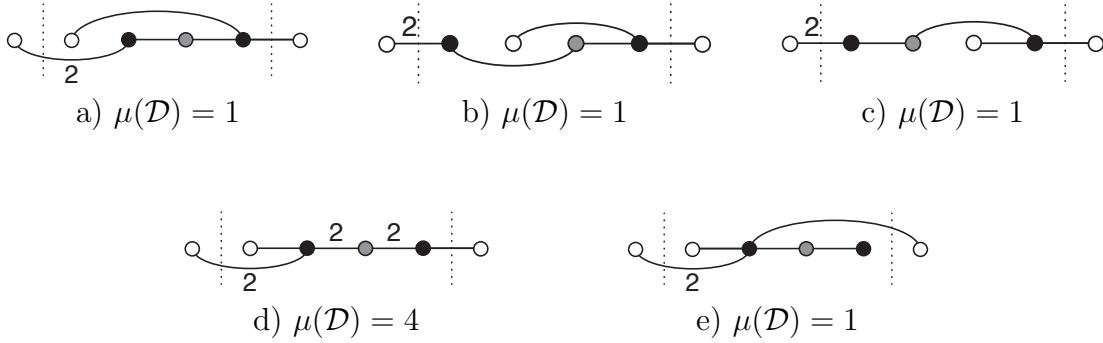


FIGURE 3. Computation of  $N_0^{(01,1,1,0)}(3, 1, 1) = 8$ .

The *cone* of  $X$  is  $\text{cone}(X) = \{\sum t_i a_i \mid t_i \geq 0\}$ . We will assume that  $X$  is *pointed* cone; *i.e.*,  $\text{cone}(X)$  does not contain a nontrivial linear subspace. This is equivalent to requiring that  $X$  lies in some open half-space of  $\mathbb{R}^d$ ; we also say  $X$  is *pointed*.

**4.1. Weighted partition functions.** If  $X = \{a_1, \dots, a_m\} \subset \mathbb{Z}^d$  is a pointed vector configuration, we define the *partition function*  $\mathcal{P}_X : \mathbb{Z}^d \rightarrow \mathbb{Z}$  to be

$$\mathcal{P}_X(c) = (\text{number of ways of writing } c = \sum_{i=1}^m c_i a_i \text{ with } c_i \in \mathbb{N}).$$

Equivalently,  $\mathcal{P}_X(c)$  is the number of integer points in the polytope:

$$P_X(c) = \{y \in \mathbb{R}^m \mid Xy = c, y \geq 0\}.$$

More generally, if  $f \in \mathbb{R}[x_1, \dots, x_m]$  is a polynomial function, we define the *weighted partition function*

$$\mathcal{P}_{X,f}(c) = \sum_{y \in P_X(c) \cap \mathbb{Z}^m} f(y).$$

We may think of this as a “discrete integral” of the function  $f$  over the polytope  $P_X(c)$ . When  $f = 1$  we recover the ordinary partition function.

We wish to know the polytope  $P_X(c)$  and the weighted partition function  $\mathcal{P}_{X,f}(c)$  vary for fixed  $X$  and variable  $c$ . As  $c$  varies, the defining hyperplanes of the polytope move, but their facet directions stay fixed. Naturally, as we push hyperplanes past vertices, we may change the combinatorial shape of the polytope. However, when the combinatorial shape is fixed, it is reasonable to expect that the discrete integral  $\mathcal{P}_{X,f}(c)$  of a polynomial  $f$  should change predictably. We now make this precise.

The *chamber complex*  $\text{Ch}(X)$  of  $X$  is a polyhedral complex supported on  $\text{cone}(X)$ . It is given by the common refinement of all the cones spanned by subsets of  $X$ .

A function  $f : \mathbb{Z}^n \rightarrow \mathbb{R}$  is *quasipolynomial* if there exists a sublattice  $\Lambda \subseteq \mathbb{Z}^n$  of full rank and polynomials  $f_1, \dots, f_N$  corresponding to the different cosets  $\Lambda_1, \dots, \Lambda_N$  of  $\Lambda$  such that  $f(v) = f_i(v)$  for every  $v \in \Lambda_i$ .

A function  $f : \mathbb{Z}^n \rightarrow \mathbb{R}$  is *piecewise quasipolynomial on  $\text{Ch}(X)$*  if the restriction of  $f$  to any given face  $F$  of the chamber complex  $\text{Ch}(X)$  is equal to a quasipolynomial function  $f^F$  depending on  $F$ .

**Theorem 4.1.** [Bla64, Stu95] *For any pointed vector configuration  $X \subset \mathbb{Z}^d$ , the partition function  $\mathcal{P}_X$  is piecewise quasipolynomial on the chamber complex  $\text{Ch}(X)$ . Furthermore, if  $X$  is unimodular, then  $\mathcal{P}_X$  is piecewise polynomial. The polynomial pieces of  $\mathcal{P}_X$  have degree  $|X| - r(X)$ .*

We will need an extension of this result. For each subset  $Y \subseteq X$  let  $\pi_Y : \mathbb{R}^m \rightarrow \mathbb{R}$  be the function  $\pi_Y(y) = \prod_{i \in Y} y_i$ . (Here we are identifying the coordinates of  $\mathbb{R}^m$  with the corresponding vectors of  $X = \{a_1, \dots, a_m\}$ .)

**Theorem 4.2.** *For any pointed vector configuration  $X \subset \mathbb{Z}^d$  and any subset  $Y \subseteq X$ , the weighted partition function  $\mathcal{P}_{X, \pi_Y}$  is piecewise quasipolynomial in the chamber complex  $\text{Ch}(X)$ . Furthermore, if  $X$  is unimodular, then  $\mathcal{P}_{X, \pi_Y}$  is piecewise polynomial. The polynomial pieces of  $\mathcal{P}_X$  have degree  $|X| + |Y| - r(X)$ .*

*Proof.* Consider the multiset  $X_Y = X \cup \{a_i : i \in Y\}$  obtained by repeating the vectors in  $Y$ . The partition function  $\mathcal{P}_{X_Y}(c)$  of this new configuration  $X_Y$  counts the ways of writing  $c = \sum_{i=1}^m c_i a_i + \sum_{i \in Y} d_i a_i$  with  $c_i, d_i \in \mathbb{N}$ .

Now note that this is the same as first writing  $c = \sum_{i=1}^m k_i a_i$  with  $k_i \in \mathbb{N}$ , and then writing  $k_i = c_i + d_i$  with  $c_i, d_i \in \mathbb{N}$  for each  $i \in Y$ . For each choice of  $(k_1, \dots, k_m)$  there are  $\prod_{i \in Y} (k_i + 1)$  of carrying out the second step. Therefore

$$\mathcal{P}_{X_Y}(c) = \sum_{k \in \mathcal{P}_X(c) \cap \mathbb{Z}^m} \prod_{i \in Y} (k_i + 1) = \sum_{T \subseteq Y} \mathcal{P}_{X, \pi_T}(c).$$

By the inclusion-exclusion formula, we get

$$(4.1) \quad \mathcal{P}_{X, \pi_Y}(c) = \sum_{T \subseteq Y} (-1)^{|Y-T|} \mathcal{P}_{X_T}(c).$$

For any  $T$ , the partition function  $\mathcal{P}_{X_T}(c)$  is quasipolynomial on each face of the chamber complex of  $X_T$ , which coincides with the chamber complex of  $X$ . It follows that the weighted partition function  $\mathcal{P}_{X, \pi_Y}(c)$  is quasipolynomial on  $\text{Ch}(X)$  as well. The second statement also follows immediately.

Finally, the claim about the degree of  $\mathcal{P}_{X, \pi_Y}(c)$  follows immediately from (4.1). However, it is also useful to give a more intuitive argument. Regard  $\mathcal{P}_{X, \pi_Y}(c)$  as the *discrete integral* of the function  $\pi_Y$  (which is polynomial of degree  $|Y|$ ) over the polytope  $P_X(c)$  (which has dimension  $d = |X| - r(X)$ ). In each face of the chamber complex, where  $P_X(c)$  has a fixed combinatorial shape, the actual integral  $\int_{P_X(c)} \pi_Y(k) dk$  is polynomial in  $c$  of degree  $d + |Y|$  by [BV97, Theorem 2.15]. Now, we can approximate this integral using increasingly fine lattices; it equals

$$\lim_{N \rightarrow \infty} \frac{1}{N^d} \sum_{k \in \mathcal{P}_X(c) \cap (\frac{1}{N}\mathbb{Z})^m} \pi_Y(k) = \lim_{N \rightarrow \infty} \sum_{l \in \mathcal{P}_X(Nc) \cap \mathbb{Z}^m} \frac{\pi_Y(l)}{N^{d+|Y|}} = \lim_{N \rightarrow \infty} \frac{\mathcal{P}_{X, \pi_Y}(Nc)}{N^{d+|Y|}}.$$

This is only possible if  $\mathcal{P}_{X, \pi_Y}(c)$  also has degree  $d + |Y| = |X| + |Y| - r(X)$ , as desired.  $\square$

**Remark 4.3.** In fact, Theorem 4.2 holds for any weighted partition function  $\mathcal{P}_{X, f}(c)$  where  $f$  is polynomial. This result is known to experts on partition functions, and

certainly not surprising in view of Theorem 4.1. Our proof of Theorem 4.2 can be adapted to prove this more general statement; for details, see [Ard09].

**Example 4.4.** The motivating example is *Kostant's partition function*  $\mathcal{P}_{A_{d-1}}(c)$  corresponding to the root system  $A_{d-1} = \{e_i - e_j : 1 \leq i < j \leq d\}$  where  $e_1, \dots, e_d$  is the canonical basis of  $\mathbb{R}^d$ . This function plays a fundamental role in the representation theory of the Lie algebra  $\mathfrak{gl}_n$ . (More generally, the representation theory of a semisimple Lie algebra is intimately related to the partition function of the corresponding root system; see [FH91, Kos59] for details.)

It is well known and not difficult to show that  $A_{d-1}$  is unimodular. In the vector space  $V_d = \{x \in \mathbb{R}^d : x_1 + \dots + x_d = 0\}$ , consider the *all-subset hyperplane arrangement* (also known as *discriminant arrangement*  $\mathcal{S}_d$  consisting of the following  $2^{d-1} - 1$  distinct hyperplanes.

$$\mathcal{S}_d : \quad \sum_{i \in S} x_i = 0 \quad (\emptyset \subsetneq S \subsetneq [d]).$$

Note that  $\sum_{i \in S} x_i = 0$  and  $\sum_{i \in [d]-S} x_i = 0$  are the same hyperplane. The root system  $A_{d-1}$  is contained in  $V_d$ , and the hyperplanes spanned by roots in  $A_{d-1}$  are precisely the hyperplanes in  $\mathcal{S}_d$ . Therefore the chamber complex of  $A_{d-1}$  in  $V_d$  is the restriction of the all-subset arrangement  $\mathcal{S}_d$  to  $\text{cone}(A_{d-1})$ .

**4.2. Weighted Ehrhart reciprocity.** Say a polytope  $P \subset \mathbb{R}^d$  is *rational* if its vertices are rational points, and *integral* if its vertices are lattice points. Let  $P^\circ$  be the *relative interior* of  $P$ ; that is, the topological interior of  $P$  inside its affine span.

**Theorem 4.5** (Ehrhart reciprocity [BR07, Ehr62]). *Let  $P$  be a rational polytope in  $\mathbb{R}^d$ , and  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a polynomial function. For each positive integer  $n$ , let*

$$L_P(n) = |nP \cap \mathbb{Z}^d|, \quad L_{P^\circ}(n) = |nP^\circ \cap \mathbb{Z}^d|.$$

*count the lattice points in the  $n$ th dilate of  $P$  and in its interior, respectively. Then  $L_P$  and  $L_{P^\circ}$  extend to quasipolynomial functions which satisfy*

$$L_{P^\circ}(x) = (-1)^{\dim P} L_P(-x).$$

*Furthermore, if  $P$  is a lattice polytope, then  $L_P$  and  $L_{P^\circ}$  are polynomial.*

The function  $L_P(x)$  is called the *Ehrhart (quasi-)polynomial* of  $P$ . We need a weighted version of this result.

**Theorem 4.6** (Weighted Ehrhart reciprocity). *Let  $P$  be a rational polytope in  $\mathbb{R}^d$ . For each positive integer  $n$ , let*

$$L_{P,f}(n) = \sum_{y \in nP \cap \mathbb{Z}^m} f(y), \quad L_{P^\circ,f}(n) = \sum_{y \in nP^\circ \cap \mathbb{Z}^m} f(y)$$

*Then  $L_{P,f}$  and  $L_{P^\circ,f}$  extend to quasipolynomial functions which satisfy*

$$L_{-P^\circ,f}(x) = (-1)^{\dim P} L_{P,f}(-x)$$

*Furthermore, if  $P$  is a lattice polytope, then  $L_P$  and  $L_{P^\circ}$  are polynomial.*

Again, experts in Ehrhart theory will probably not find this result surprising or difficult to prove, but we have only seen it stated explicitly in [Ard09] and (without proof) in [CJM11]. We will only use it for  $P = P_X(c) = \{y \in \mathbb{R}^m : Xy = c, y \geq 0\}$  and  $f = \pi_S$  for  $S \subseteq [m]$ , where  $\pi_S(y) = \prod_{i \in S} y_i$ , so we will present a proof for this case. For a proof of the general statement, see [Ard09].

*Proof of Theorem 4.6 for  $P = P_X(c)$  and  $f = \pi_S$ .* By (4.1) we have

$$(4.2) \quad L_{P_X(c), \pi_S}(n) = \mathcal{P}_{X, \pi_S}(nc) = \sum_{T \subseteq S} (-1)^{|S-T|} \mathcal{P}_{X_T}(nc) = \sum_{T \subseteq S} (-1)^{|S-T|} L_{\mathcal{P}_{X_T}(c)}(n)$$

Now we need an ‘‘interior’’ version of (4.2). Let  $\mathcal{P}_X^\circ(c)$  denote the number of ways of expressing  $c$  as a **positive** combination of vectors in  $X$  that uses all vectors in  $X$ . This is the number of lattice points in the interior  $P_X(c)^\circ$ . Also let  $\mathcal{P}_{X,f}^\circ(c)$  be the sum of  $f(y)$  over all  $y \in P_X(c)^\circ$

Note that, by the same argument we used to prove (4.1), we get

$$\mathcal{P}_{X_S}^\circ(c) = \sum_{k \in \mathcal{P}_X(c)^\circ \cap \mathbb{Z}^m} \prod_{i \in S} (k_i - 1) = \sum_{T \subseteq S} (-1)^{|S-T|} \mathcal{P}_{X, \pi_T}^\circ(c)$$

which, using the inclusion-exclusion formula gives

$$(4.3) \quad L_{P_X(c)^\circ, \pi_S}(n) = \mathcal{P}_{X, \pi_S}^\circ(nc) = \sum_{T \subseteq S} \mathcal{P}_{X_T}^\circ(nc) = \sum_{T \subseteq S} L_{P_{X_T}(c)^\circ}(n).$$

To relate (4.2) and (4.3), notice that Ehrhart reciprocity (Theorem 4.5) tells us that

$$L_{\mathcal{P}_{X_T}(c)^\circ}(n) = (-1)^{\dim P_X(c) + |T|} L_{\mathcal{P}_{X_T}(c)}(-n)$$

since  $\dim P_{X_T}(nc) = \dim P_X(c) + |T|$ . Finally it remains to notice that

$$L_{-P_{X_T}(c)^\circ, \pi_S}(n) = (-1)^{|S|} L_{P_{X_T}(c)^\circ, \pi_S}(n)$$

for all natural numbers  $n$ , and hence for all  $n$ . Combined with (4.2) and (4.3), this gives the desired result.  $\square$

## 5. PROOFS OF THEOREMS 1.3 AND 1.4

Recall that we encode the double Gromov-Witten invariants of the Hirzebruch surfaces  $\mathbb{F}_k$  in the function

$$F_{a,k,g}^{n_1, n_2}(\mathbf{x}, \mathbf{y}) = N_g^{\alpha, \beta, \tilde{\alpha}, \tilde{\beta}}(a, b, k)$$

where  $\alpha_i, \beta_i$  (resp.  $\tilde{\alpha}_i, \tilde{\beta}_i$ ) denote the number of entries of  $\mathbf{x}, \mathbf{y}$  that are equal to  $i$  (resp.  $-i$ ), and  $b = \sum i(\tilde{\alpha}_i + \tilde{\beta}_i)$ . We will need two simple lemmas.

**Lemma 5.1.** *The genus of a floor diagram  $\mathcal{D}$  is given by  $g(\mathcal{D}) = 1 - v_B + v_G$ , where  $v_B$  and  $v_G$  are the numbers of black and gray vertices, respectively.*

*Proof.* The genus of  $\mathcal{D}$  is

$$g(\mathcal{D}) = 1 - |V| + |E| = 1 - (v_B + v_W + v_G) + (e_{BW} + e_{BG})$$

where  $v_B, v_W, v_G$  denote the number of black, white, and gray vertices, respectively, and  $e_{BW}, e_{BG}$  denote the number of black-white and black-gray edges. Since every

white vertex has degree 1 we have  $e_{BW} = v_W$ . Since every gray vertex has degree 2, we have  $e_{BG} = 2v_G$ . Therefore

$$g(\mathcal{D}) = 1 - v_B + v_G$$

as desired.  $\square$

The following lemma is clear from the definitions and Lemma 5.1.

**Lemma 5.2.** *A floor diagram for  $\mathbb{F}_k$  of bidegree  $(a, b)$  and type  $(n_1, n_2)$  has:*

- *$a$  black vertices,  $g + a - 1$  gray vertices, and  $n_1 + n_2$  white vertices.*
- *$2(g + a - 1)$  black-gray edges, and  $n_1 + n_2$  black-white edges.*

We are now ready to prove our main results.

### 5.1. Proof of Theorem 1.3.

**Theorem 1.3.** *Let  $k, g, n_1, n_2 \geq 0$  and  $a \geq 1$  be fixed integers. The function*

$$F_{a,k,g}^{n_1, n_2}(\mathbf{x}, \mathbf{y})$$

*of double Gromov-Witten invariants of the Hirzebruch surface  $\mathbb{F}_k$  is piecewise polynomial in the chambers of the hyperplane arrangement*

$$\sum_{i \in S} x_i + \sum_{j \in T} y_j + rk = 0 \quad (S \subseteq [n_1], T \subseteq [n_2], 0 \leq r \leq a),$$

$$y_i - y_j = 0 \quad (1 \leq i < j \leq n_2)$$

*inside  $\Lambda = \{(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \mid \sum x_i + \sum y_i + ak = 0\} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ .*

*Proof.* By Theorem 3.4,  $F_{a,k,g}(\mathbf{x}, \mathbf{y}) = F_{a,k,g}^{n_1, n_2}(\mathbf{x}, \mathbf{y})$  is given by  $\sum_{\mathcal{D}} \mu(\mathcal{D})$  as we sum over all floor diagrams  $\mathcal{D}$  for  $\mathbb{F}_k$  having bidegree  $(a, b)$ , genus  $g$ , divergence multiplicity vector  $(\alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ , and left/right sequence  $\mathbf{x}$ . For each such floor diagram  $\mathcal{D}$ , let  $\underline{\mathcal{D}}$  be the *unweighted floor diagram* obtained by removing the weights of  $\mathcal{D}$ . We let the underlying graph  $\underline{\mathcal{D}}$  inherit the partition  $V = L \sqcup C \sqcup R$  of the vertices, the ordering of  $C$ , and the coloring of the vertices. By Lemma 5.2, the collection  $\mathcal{G}$  of underlying graphs  $\underline{\mathcal{D}}$  that may contribute to  $F_{a,k,g}(\mathbf{x}, \mathbf{y})$  is finite in number, and depends only on  $g, a$ , and  $n_1 + n_2$ .

For each graph  $G \in \mathcal{G}$ , let  $W_{G,k}(\mathbf{x}, \mathbf{y})$  be the set of weightings  $w : E(G) \rightarrow \mathbb{Z}_{>0}$  for which the resulting  $\mathcal{D}$  is a floor diagram for  $\mathbb{F}_k$  (so in particular every black vertex has divergence  $k$  and every gray vertex has divergence 0) whose white divergence sequence is  $(\mathbf{x}, \mathbf{y})$ . Note that such a  $\mathcal{D}$  automatically has genus  $g$  and bidegree  $(a, b)$ .

The multiplicity of the resulting floor diagram  $\mathcal{D}$  is  $\pi_{\text{int}}(w)$ , where  $\pi_{\text{int}} : \mathbb{R}^{E(G)} \rightarrow \mathbb{R}$  is the polynomial function defined by  $\pi_{\text{int}}(w) = \prod_{e \text{ internal}} w(e)$ . Therefore

$$(5.1) \quad F_{G,k}(\mathbf{x}, \mathbf{y}) = \sum_{w \in W_{G,k}(\mathbf{x}, \mathbf{y})} \pi_{\text{int}}(w)$$

is a contribution of  $G$  to  $F_{a,k,g}(\mathbf{x}, \mathbf{y})$ ; but it is not the only one. We need to keep in mind that  $F_{G,k}(\mathbf{x}, \mathbf{y})$  depends on the order of the entries of  $\mathbf{y}$ , while in  $F_{a,k,g}(\mathbf{x}, \mathbf{y})$  we need to consider all the distinct orders for  $\mathbf{y}$ ; see Theorem 3.4.

It follows that

$$(5.2) \quad F_{a,k,g}(\mathbf{x}, \mathbf{y}) = \frac{1}{\beta_1! \beta_2! \cdots \tilde{\beta}_1! \tilde{\beta}_2! \cdots} \sum_{G \in \mathcal{G}} \sum_{\sigma \in \mathfrak{S}_{n_2}} F_{G,k}(\mathbf{x}, \sigma(\mathbf{y}))$$

where  $\mathfrak{S}_{n_2}$  is the set of permutations of a set with  $n_2$  elements.

Now let us study the function  $F_{G,k}(\mathbf{x}, \mathbf{y})$  of (5.1) more closely. Before we proceed with the general case, let us discuss the example of Figure 4.

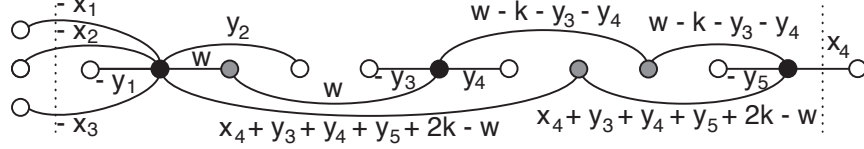


FIGURE 4. A floor diagram for the Hirzebruch surface  $\mathbb{F}_2$ .

Consider a weighting giving rise to divergences  $(\mathbf{x}, \mathbf{y})$  at the white vertices,  $k$  at the black vertices, and  $0$  at the gray vertices. The weighting is fully determined by the weight  $w$  of the edge from the first black to the first gray vertex, as shown. For this graph  $G$  to contribute to  $F_{a,k,g}(\mathbf{x}, \mathbf{y})$  in the first place, we need

$$x_1, x_2, x_3, y_1, y_3, y_5 < 0, \quad x_4, y_2, y_4 > 0.$$

Also, for the weight  $w$  to lead to a valid weighting, we need

$$w > 0, \quad w - k - y_3 - y_4 > 0 \quad x_4 + y_3 + y_4 + y_5 + 2k - w > 0.$$

Therefore  $F_{G,k}(\mathbf{x}, \mathbf{y})$  equals

$$\sum_{w=\max(0, k+y_3+y_4)}^{x_4+y_3+y_4+y_5+2k} (-y_1)y_2(-y_3)y_4(-y_5)w^2(w-k-y_3-y_4)^2(x_4+y_3+y_4+y_5+2k-w)^2.$$

If we fix the relative order of  $0, y_3 + y_4 + k$  and  $x_4 + y_3 + y_4 + y_5 + 2k$ , this function is clearly given by a fixed polynomial in  $(\mathbf{x}, \mathbf{y})$  for fixed  $a$  and  $k$ . However, this polynomial changes when we change their relative order.

In the general case, the set of weightings  $W_{G,k}(\mathbf{x}, \mathbf{y})$  that interests us is equal to the set of lattice points in a flow polytope. Given a sequence  $\mathbf{d} \in \mathbb{R}^V$ , the *flow polytope*  $\Phi_G(\mathbf{d})$  is

$$\Phi_G(\mathbf{d}) = \{w \in \mathbb{R}^{E(G)} : w_e \geq 0 \text{ for all edges } e, \text{div}(e) = \mathbf{d}_v \text{ for all vertices } v\},$$

where we think of  $w$  as a vector of flows on the edges of  $G$  and, as before, the *divergence* of a vertex is defined to be

$$\text{div}(v) = \sum_{\substack{\text{edges } e \\ v' \xrightarrow{e} v}} w_e - \sum_{\substack{\text{edges } e \\ v \xrightarrow{e} v'}} w_e.$$

The flow polytope may be rewritten in matrix form as

$$\Phi_G(\mathbf{d}) = \{w \in \mathbb{R}^{E(G)} : A\mathbf{w} = \mathbf{d}, \mathbf{w} \geq 0\},$$

where  $A \in \mathbb{R}^{V(G) \times E(G)}$  is the *adjacency matrix* of  $G$ , defined by

$$A_{v,e} = \begin{cases} 1 & \text{when } v' \xrightarrow{e} v \text{ for some } v' \\ -1 & \text{when } v \xrightarrow{e} v' \text{ for some } v' \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly  $W_{G,k}(\mathbf{x}, \mathbf{y}) = \Phi_G(\mathbf{d})$  where the entries of  $\mathbf{d}$  are given by  $(\mathbf{x}, \mathbf{y})$  for the white vertices, and are equal to  $k$  for the black vertices and 0 for the gray vertices.

From Theorem 4.2, taking into account that the columns of the adjacency matrix  $A$  are a subset of the (unimodular) root system  $A_{|E(G)|-1}$ , we obtain that the weighted partition function

$$\mathcal{P}_{G,\pi_{\text{int}}}(\mathbf{d}) = \sum_{w \in \Phi_G(\mathbf{d})} \pi_{\text{int}}(w)$$

is piecewise polynomial on the chambers of the all-subset hyperplane arrangement in  $\{\mathbf{d} \in \mathbb{R}^V : \sum_i d_i = 0\}$ . Recall that this arrangement consists of the hyperplanes  $\sum_{i \in V'} d_i = 0$  for all subsets  $V' \subseteq V$ .

We are only interested in the values of this function  $\mathcal{P}_{G,\pi_{\text{int}}}(\mathbf{d})$  on the subspace determined by the equations

$$d_u = 0 \text{ for all gray } u, \quad d_v = k \text{ for all black } v.$$

Since the sum of the divergences is 0, we have  $\sum x_i + \sum y_j + ak = 0$  automatically.

The restriction of the weighted partition function  $\mathcal{P}_{G,\pi_{\text{int}}}(\mathbf{d})$  to this subspace is the function  $F_{G,k}(\mathbf{x}, \mathbf{y})$  of (5.1). It remains piecewise polynomial, and the chamber structure is as stated. When we symmetrize in (5.2), the result  $\sum_{\sigma \in \mathfrak{S}_{n_2}} F_{G,k}(\mathbf{x}, \sigma(\mathbf{y}))$  is still piecewise polynomial in the same chambers, since the chamber structure is fixed under permutation of the  $n_2$   $\mathbf{y}$ -variables. (Note that on a face inside a wall  $y_i = y_j$ , the polynomial is smaller than the nearby polynomials for  $y_i < y_j$  and  $y_i > y_j$ , due to (5.2).) The desired result follows.  $\square$

**5.2. Proof of Theorem 1.4.** Having established the piecewise polynomiality of  $F_{a,k,g}^{n_1,n_2}(\mathbf{x}, \mathbf{y})$  by framing in terms of lattice point enumeration, we are ready to prove our next theorem. A similar argument was given in [Ard09] and [CJM11] for double Hurwitz numbers.

**Theorem 1.4.** *Each polynomial piece of  $F_{a,k,g}^{n_1,n_2}(\mathbf{x}, \mathbf{y})$  has degree  $n_2 + 3g + 2a - 2$ , and is either even or odd.*

*Proof.* In the notation of the proof of Theorem 1.3, it suffices to show these claims for the following piecewise polynomial function for each graph  $G$ :

$$F_{G,k}(\mathbf{x}, \mathbf{y}) = \sum_{w \in W_{G,k}(\mathbf{x}, \mathbf{y})} \pi_{\text{int}}(w)$$

The degree of the polynomial  $\pi_{\text{int}}(w)$  is the number of interior edges; by Lemma 5.2 this is  $n_2 + 2(g + a - 1)$ . In each full-dimensional chamber, the polytope  $W_{G,k}(\mathbf{x}, \mathbf{y})$  has

dimension  $g$ ; to see this, observe that if we fix the flow on  $g$  edges whose removal turns the graph into a tree, the whole flow vector will be uniquely determined. Clearly  $g$  is the smallest number with this property.

Repeating the argument at the end of Theorem 4.2, it follows that the polynomial pieces of  $F_{G,k}(\mathbf{x}, \mathbf{y})$  have degree  $g + [n_2 + 2(g + a - 1)]$ , which proves the second claim.

For the first claim, notice that

$$F_{G,k}(t\mathbf{x}, t\mathbf{y}) = \sum_{w \in tW} \pi_{\text{int}}(w) = L_{W, \pi_{\text{int}}}(t)$$

where we write  $W = W_{G,k}(\mathbf{x}, \mathbf{y})$ . Therefore if  $W^\circ$  denotes the relative interior of  $W$ ,

$$F_{G,k}(-t\mathbf{x}, -t\mathbf{y}) = L_{W, \pi_{\text{int}}}(-t) = (-1)^g L_{-W^\circ, \pi_{\text{int}}}(t)$$

using weighted Ehrhart reciprocity (Theorem 4.6). Recalling that the number of internal edges in  $G$  is always  $i = n_2 + 2(g + a - 1)$ , we have  $\pi_{\text{int}}(-w) = (-1)^i \pi_{\text{int}}(w)$  for any  $w \in \mathbb{R}^E(G)$ , so we get

$$F_{G,k}(-t\mathbf{x}, -t\mathbf{y}) = (-1)^{g+i} L_{W^\circ, \pi_{\text{int}}}(t)$$

and since  $\pi_{\text{int}}(w) = 0$  whenever  $w$  is in the boundary of  $W^\circ$  (which is given by equalities of the form  $w_e = 0$ ), we get

$$F_{G,k}(-t\mathbf{x}, -t\mathbf{y}) = (-1)^{g+i} L_{W, \pi_{\text{int}}}(t) = (-1)^{n_2+3g+2a-2} F_{G,k}(t\mathbf{x}, t\mathbf{y}).$$

Therefore, depending on the parity of  $n_2 + 3g + 2a - 2$ ,  $F_{G,k}(t\mathbf{x}, t\mathbf{y})$  is even or odd.  $\square$

## 6. EXAMPLE

We conclude by computing explicitly the functions  $F_{2,k,g}^{2,1}(x_1, x_2, y_1)$  for any Hirzebruch surface  $\mathbb{F}_k$  and any genus  $g$ . They are listed in Table 1; see the last paragraph of this section for the conventions used. The domain  $\Lambda = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^2 \times \mathbb{Z} : x_1 + x_2 + y_1 + 2k = 0\}$  is divided into 16 chambers, inside each one of which the function is polynomial.

These chambers are cut out by the six planes with equations

$$x_1 = 0, \quad x_1 + k = 0, \quad x_2 = 0, \quad x_2 + k = 0, \quad y_1 = 0, \quad y_1 + k = 0.$$

as shown in Figure 5. We label each chamber with a triple  $s_1 s_2 s_3$ , where each  $s_i$  is  $+$ ,  $0$ , or  $-$  according to whether the corresponding variable is greater than 0, between  $-k$  and 0, or less than  $-k$ , respectively. For example, the chamber  $+ 0 -$  is given by the inequalities

$$x_1 + k > x_1 > 0, \quad x_2 + k > 0 > x_2, \quad 0 > y_1 + k > y_1.$$

Since  $F_{2,k,g}^{2,1}(x_2, x_1, y_1) = F_{2,k,g}^{2,1}(x_1, x_2, y_1)$ , it is sufficient to compute this function for  $x_1 \geq x_2$ . For this reason, we focus on the 10 chambers intersecting the half plane  $x_1 \geq x_2$ ; the corresponding polynomials are listed in Table 1. The polynomials on the remaining 6 chambers can be obtained by symmetry.



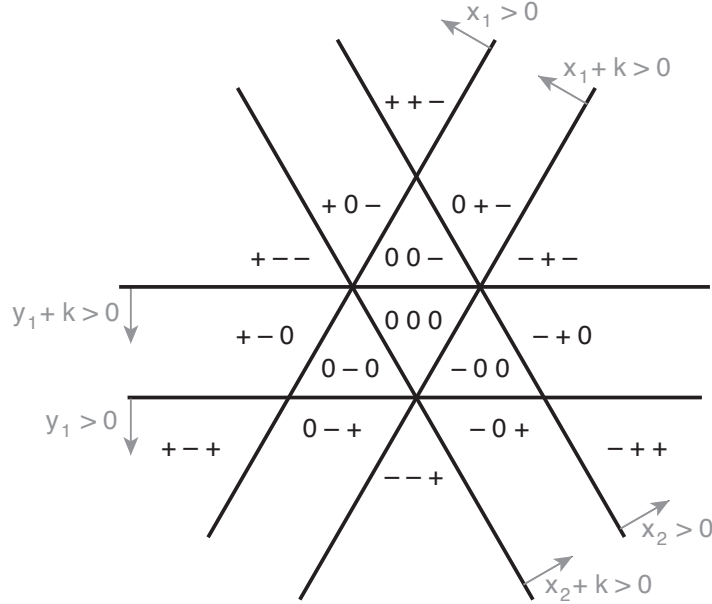


FIGURE 5. The chamber complex for  $F_{2,k,g}^{2,1}(x_1, x_2, y_1)$ .

We begin by discussing the case where the genus is  $g = 0$ . Figure 6 shows all graphs that can contribute to  $F_{2,1,0}^{2,1}(x_1, x_2, y_1)$ , obtained by a careful but straightforward case-by-case analysis.

Each graph contributes in only some of the chambers. Consider, for example, graph  $A1$ . From left to right, its edge weights must be  $-x_1, -y_1, x_2 + k, x_2 + k, x_2$  so that the vertices will have the correct divergences. Therefore  $A1$  contributes to  $F_{2,1,0}^{2,1}(x_1, x_2, y_1)$  with weight  $-y_1(x_2 + k)^2$  as long as  $x_1 < 0, y_1 < 0$ , and  $x_2 + k > 0$ ; that is, in chambers  $0 + -, - + -,$  and  $- + 0$ . Carrying out this computation for all graphs and chambers, we obtain the polynomials of Table 1 when  $g = 0$ , with  $\Gamma(w) = (w + k)^2$ .

Note that for each graph in rows from  $F$  to  $K$  (i.e. when  $x_1$  and  $x_2$  have the same sign), there are a priori two different possibilities of labeling the vertices in  $L$  or  $R$  respectively with  $q_1$  and  $q_2$ , or  $\tilde{q}_1$  and  $\tilde{q}_2$ . The two corresponding floor diagrams are the same for graphs in columns 1 and 3, and are different for graphs in columns 2 and 4 (even when  $x_1 = x_2$ , since the corresponding two vertices in  $L$  or  $R$  are not adjacent to the same vertex in  $C$ ).

For higher genus  $g$ , the computation is not much more difficult in this special case. In each graph we simply need to replace the gray vertex and its 2 incident edges by  $g + 1$  gray vertices and the corresponding  $2(g + 1)$  edges. When there is an intermediate white vertex, we simply need to decide its position among the  $g + 1$  gray vertices; there are  $g + 3$  choices. This gives rise to various factors of  $g + 3$  in Table 1. For example, in chamber  $- + -$  and genus  $g = 0$ , the graphs  $A3, C1, E1$

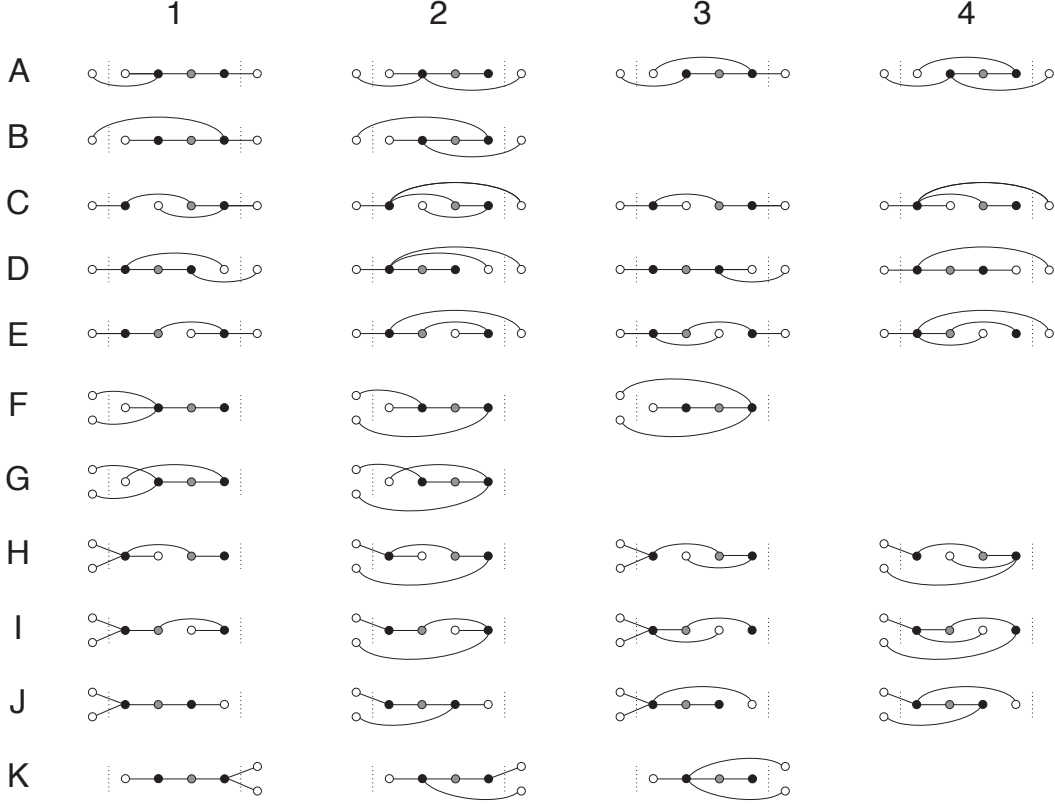


FIGURE 6. The 38 graphs that contribute to  $F_{2,1,0}^{2,1}(x_1, x_2, y_1)$ .

are isomorphic as unoriented graphs, and they account for the 3 possible positions of the white vertex in  $C$  relative to the black and gray vertices.

Suppose the two black-gray edges had weight  $w$  in a graph of genus 0. Now in the genus  $g$  graph, that total weight  $w$  has to be distributed among  $g + 1$  weights. Therefore the resulting contribution is

$$\Gamma_g(w) = \sum_{w_1 + \dots + w_{g+1} = w} \prod_{i=1}^{g+1} w_i^2,$$

where  $w_1, \dots, w_{g+1}$  are positive integers. Note that this is a polynomial of degree  $3g + 2$ , which has the same parity as  $g$ . For example we have

$$\Gamma_0(w) = w^2 \quad \text{and} \quad \Gamma_1(w) = \frac{(w-1)w(w+1)(w^2+1)}{30}.$$

To make Table 1 easier to read, we divide  $F_{w,k,g}^{2,1}(x_1, x_2, y_1)$  by  $|y_1|$  and write

$$\Gamma(w) = \Gamma_g(|w + k|).$$

## 7. CONCLUDING REMARKS

The methods exposed in this note should also be useful in other related contexts:

Chamber	Graphs ( $g = 0$ )	$F_{2,k,g}^{2,1}(x_1, x_2, y_1)/ y_1 $
0 + -	A1,A2,B1,B2	$\Gamma(x_1) + \Gamma(x_2) + \Gamma(y_1) + \Gamma(0)$
- + -	A1,A2,A3, B1,C1,E1	$(g + 3)\Gamma(x_1) + \Gamma(x_2) + \Gamma(y_1) + \Gamma(0)$
- + 0	A1,A2,A3,A4 C1,C2,E1,E2	$(g + 3)\Gamma(x_1) + \Gamma(x_2) + (g + 3)\Gamma(y_1) + \Gamma(0)$
- + +	C3,C4,D1,D2, D3,D4,E3,E4	$\Gamma(x_1) + (g + 3)\Gamma(x_2) + \Gamma(y_1) + (g + 3)\Gamma(0)$
- 0 +	H1,H2,I3,I4 J1,J2,J3,J4	$\Gamma(x_1) + (g + 3)\Gamma(x_2) + \Gamma(y_1) + (g + 3)\Gamma(0)$
- - +	H1,I3, J1, J2,J2,J3	$\Gamma(x_1) + \Gamma(x_2) + \Gamma(y_1) + (g + 3)\Gamma(0)$
+ + -	K1,K2,K2,K3	$\Gamma(x_1) + \Gamma(x_2) + \Gamma(y_1) + \Gamma(0)$
0 0 -	F1, F2,F2,F3	$\Gamma(x_1) + \Gamma(x_2) + \Gamma(y_1) + \Gamma(0)$
- 0 0	F1,F2,G1,G2, H3,H4,I1,I2	$(g + 3)\Gamma(x_1) + \Gamma(x_2) + (g + 3)\Gamma(y_1) + \Gamma(0)$
0 0 0	F1, F2F2, G1,H3,I1	$\Gamma(x_1) + \Gamma(x_2) + (g + 3)\Gamma(y_1) + \Gamma(0)$

TABLE 1. The double Gromov-Witten invariants  $F_{2,k,g}^{2,1}(x_1, x_2, y_1)$ .

- One can similarly define double Gromov-Witten invariants for more general toric surfaces. It should be possible to establish their piecewise polynomiality by pushing our method through, at least for toric surfaces corresponding to  $h$ -transverse polygons (see [BM08] for the definition of  $h$ -transversality). Using methods similar to ours, [AB13] and [LO14] prove the polynomiality of Severi degrees of many toric surfaces, including many singular ones. These papers show that Severi degrees also vary nicely as one changes the toric surface, and this might be the case for double Gromov-Witten invariants as well. (For instance, in Table 1, note that the polynomial in each chamber is also a polynomial function of  $k$ ).
- One may also try to extend this approach to *double tropical Welschinger invariants* of Hirzebruch surfaces (see [IKS09] for the definition of tropical Welschinger invariants). Due to the different treatment given to edges of even and odd weights in the real multiplicity of a floor diagram, there is no hope that double tropical Welschinger invariants are piecewise polynomial. However it is reasonable to expect that they are piecewise quasipolynomial.
- More generally, Block and Göttsche defined in [BG14b] *refined invariants* of toric surfaces. These invariants are univariate polynomials that interpolate between Gromov-Witten and tropical Welschinger invariants, and can also be

computed via floor diagrams. It would be interesting to apply the methods presented here to double refined invariants.

- It may also be possible to write explicit wall-crossing formulas describing how the function  $F_{a,k,g}^{n_1,n_2}$  changes between two adjacent chambers. For double Hurwitz numbers this was carried out in [Ard09, CJM11]; it requires additional combinatorial insight and non-trivial technical hurdles. It would be interesting to extend it to this setting.
- Floor diagrams have higher dimensional versions, at least in genus 0 (see [BM07, BM]). However one should not expect analogous piecewise polynomiality about curve enumeration in spaces of dimension at least 3. Indeed, the multiplicity of a floor diagram (i.e. the number of complex curves it encodes) includes the multiplicity of its floors. A key point for piecewise polynomiality to hold for double Hurwitz numbers and double Gromov-Witten invariants is that in dimension 1 and 2, the multiplicity of a floor is always equal to 1. However starting in dimension  $n \geq 3$ , the multiplicity of a floor is itself a Gromov-Witten invariant of a space of  $n - 1$ , and this invariant increases exponentially with the degree of the floor.

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