

The Catalan matroid.

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Abstract

We show how the set of Dyck paths of length $2n$ naturally gives rise to a matroid, which we call the “Catalan matroid” \mathbf{C}_n . We describe this matroid in detail; among several other results, we show that \mathbf{C}_n is self-dual, it is representable over \mathbb{Q} but not over finite fields \mathbb{F}_q with $q \leq n - 2$, and it has a nice Tutte polynomial.

We then generalize our construction to obtain a family of matroids, which we call “shifted matroids”. They arose independently and almost simultaneously in the work of Klivans, who showed that they are precisely the matroids whose independence complex is a shifted complex.

1 Introduction

A Dyck path of length $2n$ is a path in the plane from $(0, 0)$ to $(2n, 0)$, with steps $(1, 1)$ and $(1, -1)$, that never passes below the x -axis. It is a classical result (see for example [8, Corollary 6.2.3.(iv)]) that the number of Dyck paths of length $2n$ is equal to the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.

Each Dyck path P defines an *up-step set*, consisting of the integers i for which the i -th step of P is $(1, 1)$. The starting point of this paper is Theorem 2.1. It states that the collection of up-step sets of all Dyck paths of length $2n$ is the collection of bases of a matroid. Most of this paper is devoted to the study of this matroid, which we call the *Catalan matroid*, and denote \mathbf{C}_n .

Section 2 starts by proving Theorem 2.1. As we know, there are many equivalent ways of defining a matroid: in terms of its rank function, its

independent sets, its flats, and its circuits, among others. The rest of Section 2 is devoted to describing some of these definitions for \mathbf{C}_n .

In Section 3, we compute the Tutte polynomial of the Catalan matroid. We find that it enumerates Dyck paths according to two simple statistics. Some nice enumerative results are derived as a consequence.

In Section 4, we generalize our construction of \mathbf{C}_n to a wider class of matroids, which we call *shifted matroids*. Their name is justified by a result of Klivans, who discovered them independently, proving that they are precisely those matroids whose independence complex is a shifted complex. We then generalize our construction in a different direction to obtain, for any finite poset P and any order ideal I , a shifted family of sets. This family is not always the set of bases of a matroid.

Finally, in Section 5 we address the question of representability of the matroids we have constructed. We show that the Catalan matroid, and more generally any shifted matroid, is representable over \mathbb{Q} . In the opposite direction, we show that \mathbf{C}_n is *not* representable over the finite field \mathbb{F}_q if $q \leq n - 2$.

Throughout this paper, we will assume some familiarity with the basic concepts of matroid theory. For instance, Chapter 1 of [6] should be enough to understand most of the paper. We also highly recommend Section 6.2 and exercises 6.19-6.37 of [8] for an encyclopedic treatment of Catalan numbers and related topics.

2 The matroid

Let n be a fixed positive integer. Consider all paths in the plane which start at the origin and consist of $2n$ steps, where each step is either $(1, 1)$ or $(1, -1)$. We will call such steps *up-steps* and *down-steps*, respectively. From now on, the word *path* will always refer to a path of this form.

Such paths are in bijection with subsets of $[2n]$. To each path P , we can assign the set of integers i for which the i -th step of P is an up-step. We call this set the *up-step set* of P . Conversely, to each subset $A \subseteq [2n]$, we can assign the path whose i -th step is an up-step if and only if i is in A .

To simplify the notation later on, we will omit the brackets when we talk about subsets of $[2n]$. We will also use subsets of $[2n]$ and paths interchangeably. For example, for $n = 3$, the path 13 will be the path with up-steps at steps 1 and 3, and down-steps at steps 2, 4, 5 and 6.

A useful statistic to keep track of will be the *height of path P at x* ; i.e., the height of the path after taking its first x steps. We shall denote it $\text{ht}_P(x)$; it is equal to $2|P_{\leq x}| - x$, where $P_{\leq x}$ denotes the set of elements of P which are less than or equal to x . Also, let minht_P and maxht_P be the minimum and maximum heights that P achieves, respectively.

Theorem 2.1 *Let \mathcal{B}_n be the collection of up-step sets of all Dyck paths of length $2n$. Then \mathcal{B}_n is the collection of bases of a matroid.*

Proof. We need to check the two axioms for the collection of bases of a matroid:

(B1) \mathcal{B}_n is non-empty.

(B2) If A and B are members of \mathcal{B}_n and $a \in A - B$, then there is an element $b \in B - A$ such that $(A - a) \cup b \in \mathcal{B}_n$.

The first axiom is satisfied trivially, so we only need to check the second one. Let A and B be members of \mathcal{B}_n , and let $a \in A - B$. First we will describe those k not in A for which $A - a \cup k \in \mathcal{B}_n$, and then we will show that the smallest element of $B - A$ is one of them.

For $k \notin A$, consider the path $A - a \cup k$, which is a very slight deformation of the path A . It still consists of n up-steps and n down-steps; to determine if it is a Dyck path, we just need to check whether it goes below the x -axis. There are two cases to consider.

The first case is that $k < a$. In this case, for $k \leq c < a$, we have that $\text{ht}_{A-a \cup k}(c) = \text{ht}_A(c) + 2$. For all other values of c , we have that $\text{ht}_{A-a \cup k}(c) = \text{ht}_A(c)$. Hence the path $A - a \cup k$ stays above the path A , so it is Dyck.

The second case is that $a < k$. Here, for $a \leq c < k$, we have that $\text{ht}_{A-a \cup k}(c) = \text{ht}_A(c) - 2$. For all other values of c , we have that $\text{ht}_{A-a \cup k}(c) = \text{ht}_A(c)$. Therefore, the path $A - a \cup k$ is Dyck if and only if $\text{ht}_A(c) \geq 2$ for all $a \leq c < k$.

With that simple analysis, we can show that $A - a \cup b \in \mathcal{B}_n$, where b is the smallest element of $B - A$. If $b < a$, then we are done by the first case of our analysis. Otherwise, consider an arbitrary c with $a \leq c < b$. There are no elements of $B - A$ less than or equal to c ; so up to the c -th step, every step which is an up-step in B is also an up-step in A . Furthermore, the a -th step

is a down-step in B and an up-step in A . Therefore, $\text{ht}_A(c) \geq \text{ht}_B(c) + 2 \geq 2$. This concludes our proof. \blacksquare

A matroid is uniquely determined by its collection of bases. Therefore Theorem 2.1 defines a matroid, which will call *the Catalan matroid of rank n* (or simply *the Catalan matroid*), and denote by \mathbf{C}_n . This paper is mostly devoted to the study of this matroid.

Proposition 2.2 *The rank function of \mathbf{C}_n is given by*

$$r(A) = n + \lfloor \min \text{ht}_A / 2 \rfloor$$

for each $A \subseteq [2n]$.

Proof. Fix a subset $A \subseteq [2n]$, and let $\min \text{ht}_A = -y$, where y is a non-negative integer. Also, let x be the smallest integer such that $\text{ht}_A(x) = \min \text{ht}_A$.

Recall that the rank of a subset A of $[2n]$ is equal to the largest possible size of an intersection $A \cap B$, where B is a basis of \mathbf{C}_n .

The path A is at height $-y$ after taking $|A_{\leq x}|$ up-steps and $x - |A_{\leq x}|$ down-steps, so $|A_{\leq x}| = (x - y)/2$. Also, for any basis B , we have that $|B_{> x}| \leq n - x/2$, since $\text{ht}_B(x) \geq 0$. Hence

$$\begin{aligned} |A \cap B| &= |(A \cap B)_{\leq x}| + |(A \cap B)_{> x}| \\ &\leq |A_{\leq x}| + |B_{> x}| \leq n - y/2. \end{aligned}$$

We conclude that $r(A) \leq n + \lfloor \min \text{ht}_A / 2 \rfloor$.

Now we need a basis B with $|A \cap B| = n + \lfloor \min \text{ht}_A / 2 \rfloor$. We construct it as follows. First, add to A the smallest $a = \lceil y/2 \rceil$ numbers that it is missing, to obtain the set A' . Then $\text{ht}_{A'}(x) = 2a - y \geq 0$; in fact, it is clear that the path A' never crosses the x -axis. Let $|A| = n + h$ for some integer h ; then $\text{ht}_A(2n) = 2h$ and $\text{ht}_{A'}(2n) = 2h + 2a$. Now remove from A' the largest $h + a$ numbers that it contains, to obtain the set B . It is again easy to see that the path B never crosses the x -axis, and ends at $(2n, 0)$. So B is Dyck, and

$$|A \cap B| = |A \cap A'| - (h + a) = |A| - (h + a) = n - a$$

as desired. \blacksquare

Now that we know the rank function of \mathbf{C}_n , we can describe several important classes of subsets of the matroid. We will only provide a proof for the description of the class of flats; the remaining proofs are similar in flavor. The interested reader may want to complete the details to get better acquainted with the matroid \mathbf{C}_n .

Proposition 2.3 *The flats of \mathbf{C}_n are the subsets $A \subseteq [2n]$ such that*

(i) minht_A is odd, and

(ii) if $\text{ht}_A(x) = \text{minht}_A$, then $\{x + 1, \dots, 2n\} \subseteq A$.

Proof. Let A be a flat of \mathbf{C}_n , and let x be such that $\text{ht}_A(x) = \text{minht}_A$. If some integer y with $x + 1 \leq y \leq n$ was not in A , then we would clearly have $\text{minht}_{A \cup y} = \text{minht}_A$ and thus $r(A \cup y) = r(A)$, contradicting the assumption that A is a flat. Therefore, any flat must satisfy condition (ii).

Also, if we had a flat A with $\text{minht}_A = -2h$ achieved at $\text{ht}_A(x)$, then we would have $x \notin A$, and $\text{minht}_{A \cup x} = -2h + 1$ would be achieved at $\text{ht}_{A \cup x}(x - 1)$. We would then have $r(A \cup x) = r(A)$, again a contradiction. So any flat A must also satisfy condition (i).

Conversely, assume that A satisfies conditions (i) and (ii). Let $\text{minht}_A = -(2k + 1)$, which can only be achieved once, say at $\text{ht}_A(x)$. Any y which is not in A must be less than or equal to x ; and we have $\text{minht}_{A \cup y} = -(2k - 1)$ if $y < x$, or $\text{minht}_{A \cup y} = -2k$ if $y = x$. In either case, $r(A \cup y) = r(A) + 1$. This completes the proof. \blacksquare

Proposition 2.4 *The independent sets of \mathbf{C}_n are the subsets $A \subseteq [2n]$ such that $\text{minht}_A = \text{ht}_A(2n)$.*

Proposition 2.5 *The spanning sets of \mathbf{C}_n are the subsets $A \subseteq [2n]$ such that $\text{minht}_A = 0$.*

Proposition 2.6 *The circuits of \mathbf{C}_n are the subsets $A \subseteq [2n]$ of the form $A = \{2k, 2k + b_1, \dots, 2k + b_{n-k}\}$, for some positive integer $k \leq n$ and some Dyck path $\{b_1, \dots, b_{n-k}\}$ of length $2(n - k)$.*

Proposition 2.7 *The bonds of \mathbf{C}_n are the subsets $A \subseteq [2n]$ such that*

(i) $\text{maxht}_A = 1$, and

(ii) if $\text{ht}_A(x) = 1$, then A has no elements greater than x .

We complete this section with an observation which is interesting in itself, and will also be important to us in section 3.

Proposition 2.8 *The Catalan matroid is self-dual.*¹

Proof. Say $B = \{b_1, \dots, b_n\}$ is a basis of \mathbf{C}_n , and let $[2n]-B = \{c_1, \dots, c_n\}$ be the corresponding basis of the dual matroid \mathbf{C}_n^* . Then $\{2n+1-c_n, \dots, 2n+1-c_1\}$ is a Dyck path; in fact, it is the path obtained by reflecting the Dyck path B across a vertical axis. So the bases of \mathbf{C}_n^* are simply the up-step sets of all Dyck paths of length $2n$, under the relabeling $x \rightarrow 2n+1-x$. Thus $\mathbf{C}_n^* \cong \mathbf{C}_n$. ■

3 The Tutte polynomial

Given a matroid M over a ground set S , its Tutte polynomial is defined as:

$$T_M(q, t) = \sum_{A \subseteq S} (q-1)^{r(S)-r(A)} (t-1)^{|A|-r(A)}.$$

For our purposes, it is more convenient to define the Tutte polynomial in terms of the internal and external activity of the bases. We recall this definition now.

We first need to fix an arbitrary linear ordering of S .

For any basis B and any element $e \notin B$, the set $B \cup e$ contains a unique circuit. If e is the smallest element of that circuit with respect to our fixed linear order, then we say that e is *externally active* with respect to B . The number of externally active elements with respect to B is called the *external activity* of B ; we shall denote it by $e(B)$.

Dually, for any basis B and any element $i \in B$, the set $S - B \cup i$ contains a unique bond. If i is the smallest element of that bond, then we say that i is *internally active* with respect to B . The number of internally active elements with respect to B is called the *internal activity* of B ; we shall denote it by $i(B)$.²

¹We follow Oxley [6] in calling a matroid M *self-dual* if $M \cong M^*$. It is worth mentioning, however, that some authors reserve the term ‘self-dual’ for matroids M such that $M = M^*$.

²The internally active elements with respect to a basis B of M are precisely the externally active elements with respect to the basis $S - B$ of the dual matroid M^* . That is why we say that internal activity and external activity are dual concepts.

Proposition 3.1 (Crapo, [1]) *For any matroid M and any linear order of its ground set,*

$$T_M(q, t) = \sum_{B \text{ basis}} q^{i(B)} t^{e(B)}.$$

We will use Proposition 3.1 to study the Tutte polynomial of the Catalan matroid. The first thing to do is to fix a linear order of its ground set, $[2n]$. We will use the most natural choice: $1 < 2 < \dots < 2n$. Now we compute the internal and external activity of each basis of \mathbf{C}_n .

Lemma 3.2 *The internal activity of a Dyck path B is equal to the number of up-steps that B takes before its first down-step.*

Proof. Let $i \in B$. The path $[2n] - B$ never goes above height 0; the path $[2n] - B \cup i$ goes up to height 2. Let j be the smallest integer such that $\text{ht}_{[2n] - B \cup i}(j) = 1$. Clearly $j \geq i$.

Let D be the unique bond of \mathbf{C}_n which can be obtained by deleting some elements of $[2n] - B \cup i$. We cannot delete any element less than or equal to j , or else the resulting path will not reach height 1. We must delete any element larger than j by Proposition 2.7. So $D = ([2n] - B)_{\leq j}$.

Therefore, i is the smallest element of D if and only if B contains all of $1, 2, \dots, i - 1$. This completes the proof. \blacksquare

Lemma 3.3 *The external activity of a Dyck path B is equal to the number of positive integers x for which $\text{ht}_B(x) = 0$.*

Proof. Let $e \notin B$. The path $B \cup e$ ends at height 2; let $2k - 1$ be the largest integer such that $\text{ht}_{B \cup e}(2k - 1) = 1$. Clearly $2k - 1 < e$.

We start by showing that the unique circuit C of \mathbf{C}_n contained in $B \cup e$ is $(B \cup e)_{\geq 2k}$.

Since $C \subseteq B \cup e$, we have that $\text{ht}_C(2n) - \text{ht}_C(2k - 1) \leq \text{ht}_{B \cup e}(2n) - \text{ht}_{B \cup e}(2k - 1) = 1$. Equality holds if and only if every up-step of $B \cup e$ after the $(2k - 1)$ -th is also an up-step of C ; *i.e.*, when $(B \cup e)_{\geq 2k} = C_{\geq 2k}$.

But it is clear from Proposition 2.6 that $\text{ht}_C(2n) - \min \text{ht}_C = 1$, and that $\min \text{ht}_C$ is only achieved at $\text{ht}_C(\min C - 1)$. So the above inequality can only hold if $\min C = 2k$. Thus $C = C_{\geq 2k} = (B \cup e)_{\geq 2k}$ as desired.

Now we know that $\min C = 2k$, so e is externally active if and only if $e = 2k$. If $\text{ht}_B(e) = 0$, this is clearly the case. On the other hand, if $\text{ht}_B(e) \geq 1$, then $\text{ht}_{B \cup e}(e - 1) = \text{ht}_B(e - 1) \geq 2$, so this is not the case. This completes the proof. \blacksquare

Theorem 3.4 For a Dyck path P , let $a(P)$ denote the number of up-steps that P takes before its first down-step, and let $b(P)$ denote the number of positive integers x for which $\text{ht}_P(x) = 0$

Then the Tutte polynomial of the Catalan matroid \mathbf{C}_n is equal to

$$T_{\mathbf{C}_n}(q, t) = \sum_{P \text{ Dyck}} q^{a(P)} t^{b(P)},$$

where the sum is over all Dyck paths of length $2n$.

Proof. This follows immediately from Proposition 3.1 and Lemmas 3.2 and 3.3. ■

Corollary 3.5 The polynomial

$$\sum_{P \text{ Dyck}} q^{a(P)} t^{b(P)},$$

is symmetric in q and t .

Proof. It is well-known that, for any matroid M , we have $T_{M^*}(q, t) = T_M(t, q)$. The result follows from Proposition 2.8 and Theorem 3.4. ■

It is a known fact that the statistics $a(P)$ and $b(P)$ are equidistributed over the set of Dyck paths of length $2n$. The number of paths with $a(P) = k$ and the number of paths with $b(P) = k$ are both equal to $\frac{k}{2n-k} \binom{2n-k}{n}$. For the first equality, see for example [9]; for the second, see [5, equation (7)].

Corollary 3.5 was also discovered independently by James Haglund [3]. It is not difficult to prove it directly; in fact, it will be an immediate consequence of our next theorem.

Theorem 3.6 Let $C(x) = \frac{1 - \sqrt{1 - 4x}}{2} = C_0 + C_1x + C_2x^2 + \dots$ be the generating function for the Catalan numbers. Then

$$\sum_{n \geq 0} T_{\mathbf{C}_n}(q, t) x^n = \frac{1 + (qt - q - t)x C(x)}{1 - qtx + (qt - q - t)x C(x)}.$$

Proof. A Dyck path P of length $2n \geq 2$ can be decomposed uniquely in the standard way: it starts with an up-step, then it follows a Dyck path P_1 of length $2r$, then it takes a down-step, and it ends with a Dyck path P_2

of length $2s$, for some non-negative integers r, s with $r + s = n - 1$. More precisely, and necessarily more confusingly,

$$P = \{1, 1 + p_1, 1 + p_2, \dots, 1 + p_r, 2r + 2 + q_1, 2r + 2 + q_2, \dots, 2r + 2 + q_s\}$$

for some Dyck paths $\{p_1, \dots, p_r\}$ and $\{q_1, \dots, q_s\}$ with $r + s = n - 1$.

It is clear that in this decomposition we have $a(P) = a(P_1) + 1$ and $b(P) = b(P_2) + 1$. Therefore

$$\begin{aligned} T_{\mathbf{C}_n}(q, t) &= \sum_{r+s=n-1} \sum_{P_1 \in \mathcal{B}_r} \sum_{P_2 \in \mathcal{B}_s} q^{a(P_1)+1} t^{b(P_2)+1} \\ &= qt \sum_{r+s=n-1} T_{\mathbf{C}_r}(q, 1) T_{\mathbf{C}_s}(1, t) \end{aligned}$$

for $n \geq 1$; so if we write $\mathbf{T}(q, t, x) = \sum_{n \geq 0} T_{\mathbf{C}_n}(q, t) x^n$, we have

$$\mathbf{T}(q, t, x) = 1 + qtx \mathbf{T}(q, 1, x) \mathbf{T}(1, t, x). \quad (1)$$

Now observe that $\mathbf{T}(1, 1, x) = C(x)$. Setting $q = 1$ in (1) gives a formula for $\mathbf{T}(1, t, x)$, and setting $t = 1$ gives a formula for $\mathbf{T}(q, 1, x)$. Substituting these two formulas back into (1), we get the desired result. \blacksquare

4 Shifted matroids

We now generalize our construction of \mathbf{C}_n to a larger family of matroids, which we call *shifted matroids*. There is one shifted matroid for each non-empty set $S = \{s_1 < \dots < s_n\}$ of positive integers, which we shall denote $\mathbf{SM}(s_1, \dots, s_n)$

Theorem 4.1 *Let $S = \{s_1 < \dots < s_n\}$ be a set of positive integers, and let \mathcal{B}_S be the collection of sets $\{a_1 < \dots < a_n\}$ such that $a_1 \leq s_1, \dots, a_n \leq s_n$. Then \mathcal{B}_S is the collection of bases of a matroid $\mathbf{SM}(s_1, \dots, s_n)$.*

Proof. Once again, as in the proof of Theorem 2.1, axiom **(B1)** is trivial, since $S \in \mathcal{B}_S$. We need to check axiom **(B2)**. Let $A = \{a_1 < \dots < a_n\}$ and $B = \{b_1 < \dots < b_n\}$ be in \mathcal{B}_S , and let $a_x \in A - B$. We claim that, if b_y is the smallest element in $B - A$, then $A - a_x \cup b_y \in \mathcal{B}_S$.

Let i be the integer such that $a_i < b_y < a_{i+1}$. (If $b_y < a_1$ then the claim is trivially true, since we are replacing a_x in A with a number smaller than

it. If $b_y > a_n$ then set $i = n$.) We may assume that $i \geq x$; if that was not the case, then we would have $b_y < a_{i+1} \leq a_x$, and the claim would be trivial. We then have

$$A - a_x \cup b_y = \{a_1 < \cdots < a_{x-1} < a_{x+1} < \cdots < a_i < b_y < a_{i+1} < \cdots < a_n\}$$

and we have n inequalities to check.

The first $x - 1$ and the last $n - i$ of these do not require any extra work: we already know that $a_k \leq s_k$ for $1 \leq k \leq x - 1$ and for $i + 1 \leq k \leq n$.

For each value of k with $x \leq k \leq i - 1$, we need to check that $a_{k+1} \leq s_k$. If $k \geq y$, we have $a_{k+1} \leq a_i < b_y \leq b_k \leq s_k$. Otherwise, if $k < y$, proceed as follows. Since b_y is the smallest element of B which is not in A , and a_x is not in B , the numbers b_1, \dots, b_k must all be somewhere in the list $a_1, \dots, a_{x-1}, a_{x+1}, \dots, a_i$. Therefore the k -th smallest number of this list, a_{k+1} , must be less than or equal to b_k , which is less than or equal to s_k .

Finally, we need to check that $b_y \leq s_i$. Since the numbers b_1, \dots, b_{y-1} all appear in the list $a_1, \dots, a_{x-1}, a_{x+1}, \dots, a_k, \dots, a_i$, we have $y - 1 \leq i - 1$. Therefore $b_y \leq s_y \leq s_i$. \blacksquare

A path $\{a_1 < \cdots < a_n\}$ is Dyck if and only if, for each i with $1 \leq i \leq n$, the i -th up-step comes before the i -th down-step; that is, if and only if $a_i \leq 2i + 1$. Therefore, the Catalan matroid \mathbf{C}_n is exactly the shifted matroid $\mathbf{SM}(1, 3, 5, \dots, 2n - 1)$, with an additional loop $2n$.

Recall that an *abstract simplicial complex* Δ on $[n]$ is a family of subsets of $[n]$ (called *faces*) such that if $G \in \Delta$ and $F \subseteq G$, then $F \in \Delta$. A simplicial complex Δ is *shifted* if, for any face $F \in \Delta$ and any pair of elements $i < j$ such that $i \notin F$ and $j \in F$, the subset $F - j \cup i$ is also a face of Δ .

The family of independent sets of a matroid M is always a simplicial complex, called the *independence complex* of M . For shifted matroids, we have the following simple observation.

Proposition 4.2 *The independence complex of a shifted matroid $\mathbf{SM}(s_1, \dots, s_n)$ is a shifted complex.*

Proof. If $F \subseteq [s_n]$ is independent, it is contained in some basis B . Now assume that we have two elements $i < j$ such that $i \notin F$ and $j \in F$, and let $G = F - j \cup i$. If the basis B contains i , then it contains G . Otherwise, $B - j \cup i$ is also a basis: for any $1 \leq k \leq n$, its k -th smallest is less than

or equal to the k -th smallest element of B , which is less than or equal to s_k . This basis contains G . In both cases, we conclude that G is independent. ■

In [4], Klivans characterizes *shifted matroid complexes*: shifted complexes which are the independence complex of a matroid. Her result and ours were discovered almost simultaneously. When we sat down to discuss them, we realized that the matroids that arise in her characterization are precisely the ones in our construction. This is why they were baptized “shifted matroids”.

Proposition 4.3 (Klivans, [4]) *If the independence complex of a loop-less matroid M is a shifted complex, then $M \cong \mathbf{SM}(s_1, \dots, s_n)$ for some positive integers $s_1 < \dots < s_n$.*

Theorem 4.1 and Propositions 4.2 and 4.3 have a nice application to Young tableaux. Recall that a *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ of n is a weakly decreasing sequence of positive integers which add up to n . We associate to it a *Young diagram*: a left-justified array of unit squares, which has λ_i squares on the i -th row from top to bottom.³ A *standard Young tableau* is a placement of the integers $1, \dots, n$ in the squares of the Young diagram, in such a way that the numbers are increasing from left to right and from top to bottom.

These definitions will be sufficient for our purposes. For a much deeper treatment of the theory of Young tableaux, we refer the reader to [2].

Corollary 4.4 *Let λ be a partition. Define the first row set of a standard Young tableau T of shape λ to be the set of entries which appear in the first row of T . Then the collection of first row sets of all standard Young tableaux of shape λ is the collection of bases of a shifted matroid.*

Proof. Let $\lambda' = (\lambda'_1, \dots, \lambda'_n)$ be the conjugate partition of λ , so λ'_i is the number of squares on the i -th column of the Young diagram of λ . Let $s_i = 1 + \lambda'_1 + \dots + \lambda'_{i-1}$ for $1 \leq i \leq n$.

Let $\{b_1 < \dots < b_n\}$ be the first row set of a standard Young tableau T of shape λ . The first entry on the i -th column of T is b_i ; it is smaller than every entry to its southeast. There are only $\lambda'_1 + \dots + \lambda'_{i-1}$ cells which are not to its southeast, so $b_i \leq s_i$.

³This is the English way of drawing Young diagrams; francophones draw them with λ_i squares on the i -th row from bottom to top.

Conversely, if $B = \{b_1 < \dots < b_n\}$ is such that $b_i \leq s_i$ for $1 \leq i \leq n$, then we can construct a standard Young tableau with first row set B . To do it, we first put the elements of B in order on the first row of λ . Then we put the remaining numbers from 1 to $|\lambda|$ on the remaining cells going in order down the columns, starting with the leftmost column. The inequalities $b_i \leq s_i$ guarantee that this process does indeed give a Young tableau T .

It follows that the collection in question is simply the collection of bases of the matroid $\mathbf{SM}(s_1, \dots, s_n)$. \blacksquare

We might try to generalize Corollary 4.4, replacing the first row of λ by any partition $\mu \subseteq \lambda$. Define the μ -set of a standard Young tableau T of shape λ to be the set of entries which appear in the sub-shape μ in T .

It is not too difficult to see that we do not always get the collection of bases of a matroid with this construction. However, we can still say something interesting.

Proposition 4.5 *Let $\mu \subseteq \lambda$ be partitions. Then the collection $\mathcal{B}_{\lambda\mu}$ of μ -sets of all standard Young tableau of shape λ is a shifted family.*

Proof. In fact, we prove something more general. Let P be a partially ordered set, or *poset*, of n elements. Recall that a subset I of P is an *order ideal* of P if, for any pair of elements $x, y \in P$ with $x <_P y$ and $y \in I$, we also have $x \in I$. Also recall that a *linear extension* of P is a bijection $f : P \rightarrow [n]$ such that $i <_P j$ implies that $f(i) < f(j)$. For more information on posets, we refer the reader to [7, Chapter 3].

Define the I -set of a linear extension f of P to be the set $\{f(i) : i \in I\}$.

Proposition 4.6 *Let P be a poset of n elements, and let I be an order ideal of P . Then the collection $\mathcal{B}_{P,I}$ of I -sets of all linear extensions of P is a shifted family.*

Proof of Proposition 4.6. We need to check that if we have a set $B \in \mathcal{B}_{P,I}$ and a pair of numbers $a < b$ such that $a \notin B$ and $b \in B$, then $B - b \cup a \in \mathcal{B}_{P,I}$. It is enough to show this for $a = b - 1$; the general case will then follow by induction on $b - a$.

So let f be a linear extension of P with I -set B , and let $b \in B$ be such that $b - 1 \notin B$. Let $b = f(i)$ and $b - 1 = f(p)$ where $i \in I$ and $p \in P - I$. Let $g : P \rightarrow [n]$ be defined by switching the values of f at i and p ; *i.e.*,

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \{i, p\} \\ b - 1 & \text{if } x = i \\ b & \text{if } x = p \end{cases} \quad (2)$$

We claim that g is also a linear extension for P . An important observation is that i and p are incomparable in P . If we had $i < p$, then we would have $b = f(i) < f(p) = b - 1$. If we had $i > p$, then $i \in I$ would imply $p \in I$.

To check that f is a linear extension, we need to check that $f(i) = b$ satisfies several inequalities: it must be greater than all the values that f takes on $P_{<i}$, and less than all the values that f takes on $P_{>i}$. But b is never compared to $b - 1$ here, since p and i are incomparable. Therefore, $b - 1$ also satisfies all those inequalities that b needs to satisfy.

Similarly, $b - 1$ must be greater than all the values that f takes on $P_{<p}$ and less than all the values that f takes on $P_{>p}$. The number b also satisfies these inequalities.

So we can switch the values of $f(i)$ and $f(p)$, and the resulting function g will also be a linear extension of P . Also, the I -set of g is $B - b \cup (b - 1)$. This concludes the proof. ■

Now, to prove Proposition 4.5, partially order the cells of λ : cell i is less than cell j in P_λ if and only if cell i is northeast of cell j in λ . The cells of μ define an order ideal I_μ of this poset P_λ , and $\mathcal{B}_{\lambda\mu} = \mathcal{B}_{P_\lambda, I_\mu}$. Now use Proposition 4.6. ■

5 Representability

A natural question to ask is whether the Catalan matroid can be represented as the vector matroid of a collection of vectors. We answer that question in this section.

Given a collection of real numbers x_1, \dots, x_k , let $x_S = \prod_{i \in S} x_i$ for each subset $S \subseteq [k]$. Form all the 2^{2^k} possible sums of some of the x_S 's. If these sums are all distinct, we will say that the initial collection of numbers is *generic*. Most collections of real numbers are generic. A specific example is a set of algebraically independent real numbers. Another example is any sequence of positive integers which increases quickly enough; for instance, one that satisfies $x_i > (1 + x_1)(1 + x_2) \cdots (1 + x_{i-1})$ for $1 < i \leq k$.

Theorem 5.1 *Let v_1, \dots, v_{2n} be the columns of a matrix*

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & \dots & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & a_{n6} & \dots & a_{n,2n-1} & 0 \end{pmatrix}$$

where the a_{ij} 's with $1 \leq i \leq n$ and $1 \leq j \leq 2i - 1$ are generic integers. Then the vector matroid of $\{v_1, \dots, v_n\}$ is isomorphic to the Catalan matroid \mathbf{C}_n .

Proof. Let M be the vector matroid of V . Let $1 \leq b_1 < \dots < b_n \leq 2n$. The set $B = \{v_{b_1}, \dots, v_{b_n}\}$ is a basis for M if and only if it is independent; that is, if and only if the determinant of the matrix A_B with columns v_{b_1}, \dots, v_{b_n} is non-zero.

This determinant is a sum of $n!$ terms, with plus or minus signs attached to them. Since the a_{ij} 's are generic, this sum can only be zero if all the terms are 0. So B is a basis as long as at least one of the $n!$ terms in this determinant is non-zero.

The question is now whether it is possible to place n non-attacking rooks on the non-zero entries of A_B ; that is, to choose n non-zero entries with no two on the same row or column. The marriage theorem [10, Theorem 5.1] would be the standard tool to attack this kind of question. However, A_B is such that any entry below or to the left of a non-zero entry is also non-zero. This fact will make our argument shorter and self-contained.

If $b_i \leq 2i - 1$ for all integers i with $1 \leq i \leq n$, then the (i, i) entry of A_B is $a_{i,b_i} \neq 0$. Therefore we can place n non-attacking rooks on non-zero entries of A_B by putting them on the main diagonal.

Conversely, suppose that we have a placement of n non-attacking rooks on non-zero entries of A_B . Let i be any integer between 1 and n . Then the rooks on the first i rows must be on i different columns. From the shape that the non-zero entries of A_B form, we conclude that the i -th row must contain i different non-zero entries. Thus the (i, i) entry of A_B , which is precisely a_{i,b_i} , must be non-zero. Therefore $b_i \leq 2i - 1$. \blacksquare

The above proof generalizes immediately to any shifted matroid $\mathbf{SM}(s_1, \dots, s_n)$.

Theorem 5.2 *Let $s_1 < \dots < s_n$ be arbitrary positive integers. Let v_1, \dots, v_{s_n} be the columns of a matrix $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq s_n}$, where the a_{ij} 's with $1 \leq$*

$i \leq n$ and $1 \leq j \leq s_i$ are generic, and the remaining a_{ij} 's are equal to 0. Then the vector matroid of $\{v_1, \dots, v_{s_n}\}$ is isomorphic to the shifted matroid $\mathbf{SM}(s_1, \dots, s_n)$.

Theorem 5.1 shows that the Catalan matroid is representable over \mathbb{Q} , or even over a sufficiently large finite field. In the other direction, we now show a negative result about representing \mathbf{C}_n over finite fields.

Proposition 5.3 *The Catalan matroid \mathbf{C}_n is not representable over the finite field \mathbb{F}_q if $q \leq n - 2$.*

Proof. It is known ([6], Proposition 6.5.2) and easy to show that the uniform matroid $U_{2,k}$ is \mathbb{F}_q -representable if and only if $q \geq k - 1$. A matroid containing it as a minor is not representable over \mathbb{F}_q for $q \leq k - 2$. This suggests that we should find the largest k for which $U_{2,k}$ is a minor of \mathbf{C}_n .

We can use the Scum theorem (Higgs, [6], Proposition 3.3.7), which essentially says that, if a matroid has a certain minor, then it must have that minor hanging from the top of its lattice of flats. Our question is then equivalent to finding the largest k for which there exists a rank- $(n - 2)$ flat which is contained in k rank- $(n - 1)$ flats.

Lemma 5.4 *Let A be a rank- $(n - 2)$ flat, and let x be the smallest integer such that $\text{ht}_A(x) = -1$. Then there are exactly $\frac{x+3}{2}$ rank- $(n - 1)$ flats containing A .*

Proof of Lemma 5.4. We know from Propositions 2.2 and 2.3 that $\text{minht}_A = -3$ and that, once the path A reaches height -3 , say at $\text{ht}_A(y)$, it only takes up-steps. We want to add elements to A to obtain a path which reaches a minimum height -1 , and only takes up-steps after that.

Say that we add one element a to A . This new up-step at a comes before the y -th, so $\text{ht}_{A \cup a}(y) = -1$. If we don't want to add any more elements to A , we have to make sure that $A \cup a$ only reaches height -1 at y . For this to be true, we need the new up-step a to occur on or before the x -th step. In A , there are $\frac{x+1}{2}$ down-steps up to the x -th to choose from. Each one of these gives a rank- $(n - 1)$ flat containing A .

On the other hand, if we are to add more elements to A to obtain a rank- $(r - 1)$ flat B , they will all be less than y so we will have $\text{ht}_B(y) > 0$. The minimum height in B must then be achieved at some z for which $\text{ht}_A(z) = -1$.

In fact, for this z to be unique, it must be the leftmost one, *i.e.*, it must be x . So the only possibility is that $B = A_{\leq x} \cup \{x+1, \dots, 2n\}$, which is indeed a rank- $(n-1)$ flat. This concludes the proof of Lemma 5.4. ■

Having shown Lemma 5.4, the rest is easy. The rank- $(n-2)$ flat which is contained in the largest number of rank- $(n-1)$ flats, is the latest one to arrive to height -1 . This flat is clearly $\{1, 2, \dots, n-3, n-2, 2n\}$, which arrives to height -1 after $2n-3$ steps. It is contained in exactly n rank- $(n-1)$ flats.

Therefore \mathbf{C}_n contains $U_{2,n}$ as a minor, and thus it is not representable over a field \mathbb{F}_q with $q \leq n-2$. ■

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