Enumerative and algebraic aspects of matroids and hyperplane arrangements

by

Federico Ardila

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Author . . . .
Department of Mathematics
December 18, 2002

Certified by . . . .
Richard P. Stanley
Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Rodolfo Rubén Rosales
Chairman, Applied Mathematics Committee

Accepted by . . . .
Pavel Etingof
Chairman, Department Committee on Graduate Students

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Abstract

This thesis consists of three projects on the enumerative and algebraic properties of matroids and hyperplane arrangements. In particular, a central object of study is the Tutte polynomial, which stores much of the enumerative information of these objects.

The first project is the study of the Tutte polynomial of an arrangement and, more generally, of a semimatroid. It has two components: an enumerative one and a matroid-theoretic one.

We start by considering purely enumerative questions about the Tutte polynomial of a hyperplane arrangement. We introduce a new method for computing it, which generalizes several known results. We apply our method to several specific arrangements, thus relating the computation of Tutte polynomials to problems in enumerative combinatorics. As a consequence, we obtain several new results about classical combinatorial objects such as labeled trees, Dyck paths, semiorders and alternating trees.

We then address matroid-theoretic aspects of arrangements and their Tutte polynomials. We start by defining semimatroids, a class of objects which abstracts the dependence properties of an affine hyperplane arrangement. After discussing these objects in detail, we define and investigate their Tutte polynomial. In particular, we prove that it is the universal Tutte-Grothendieck invariant for semimatroids, and we give a combinatorial interpretation for its non-negative coefficients.

The second project is the beginning of an attempt to study the Tutte polynomial from an algebraic point of view. Given a matroid representable over a field of characteristic zero, we construct a graded algebra whose Hilbert-Poincaré series is a simple evaluation of the Tutte polynomial of the matroid. This construction is joint work with Alex Postnikov.

The third project involves a class of matroids with very rich enumerative properties. We show how the set of Dyck paths of length $2n$ naturally gives rise to a matroid, which we call the Catalan matroid $\mathbf{C}_n$. We describe this matroid in detail; among several other results, we show that $\mathbf{C}_n$ is self-dual, it is representable over the rationals but not over finite fields $\mathbb{F}_q$ with $q \leq n-2$, and it has a nice Tutte polynomial. We then introduce a more general family of matroids, which we call shifted matroids. They are precisely the matroids whose independence complex is a shifted simplicial complex.

Thesis Supervisor: Richard P. Stanley
Title: Professor of Mathematics

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Chapter 1

Introduction

This thesis is concerned with enumerative and algebraic properties of matroids and hyperplane arrangements. In particular, a central object of study is the Tutte polynomial, which stores much of the enumerative information of these objects.

The *Tutte polynomial* was first introduced by Tutte [62] in the context of graphs, and later generalized to matroids by Crapo [20]. It arises very naturally in numerous enumerative problems in both of these areas.

Much of the interest in the Tutte polynomial derives from the fact that it is the universal *Tutte-Grothendieck*, or *T-G invariant*. Roughly speaking, this means that any generalized T-G invariant; *i.e.*, any invariant satisfying a deletion-contraction recursion, is an evaluation of it. This theorem is of great power and applicability: a very large number of the objects of enumerative interest in graph and matroid theory are counted by T-G invariants. Some of the generalized T-G invariants of a graph are: the number of spanning trees, the chromatic polynomial, the number of nowhere zero t-flows, and the number of acyclic orientations. Some of the generalized T-G invariants of a matroid are: the number of bases, the number of independent sets, the characteristic polynomial, the number of nbc-bases and the shelling polynomial. If we know the Tutte polynomial of a graph or a matroid, we can compute all of these invariants immediately.

This thesis consists of three projects in enumerative and algebraic combinatorics where the Tutte polynomial plays an important role. Different readers may be interested in different parts of this thesis. For that reason, we have tried to arrange and write the four chapters in such a way that they can be read independently. The work of Chapter 2 is mostly in enumerative combinatorics, Chapter 3 is in matroid theory, Chapter 4 is in algebraic combinatorics, and Chapter 5 lies at the intersection between matroid-theoretic, enumerative and topological combinatorics. We have included a more detailed introduction at the beginning of each chapter to cover some of the background material and outline the goals of the chapter.

The first project we are concerned with is the study of the Tutte polynomial of an arrangement and, more generally, of a semimatroid. It has two components: an enumerative one and a matroid-theoretic one. We have tried to keep these two parts separate in Chapters 2 and 3.

Chapter 2 is devoted to purely enumerative questions on the Tutte polynomial of a hyperplane arrangement. We introduce a new method for computing it, which generalizes several known results. We apply our method to several specific arrangements, thus relating the computation of Tutte polynomials to some problems in enumerative combinatorics. As a consequence, we obtain new results about various classical combinatorial objects.

Chapter 3 addresses matroid-theoretic aspects of arrangements and their Tutte polynomials. It starts by defining *semimatroids*, a class of objects which abstracts the dependence properties of an affine hyperplane arrangement. After analyzing in detail their structure, we define and investigate their Tutte polynomial. In particular, we prove that it is the universal Tutte-Grothendieck invariant for semimatroids, and we give a combinatorial interpretation for its non-negative coefficients.

Chapter 4 is the beginning of a project which attempts to study the Tutte polynomial from an algebraic point of view. Given a matroid representable over a field of characteristic zero, we construct a graded algebra whose Hilbert-Poincaré series is a simple transformation of the shelling polynomial of the matroid. The work of this chapter is joint work with Alex Postnikov.

Finally, Chapter 5 is devoted to the study of a class of matroids with very rich enumerative properties. We show how the set of Dyck paths of length $2n$ naturally gives rise to a matroid, which we call the *Catalan matroid* $\mathbf{C}_n$. We describe this matroid in detail; among several other results, we show that $\mathbf{C}_n$ is self-dual, it is representable over $\mathbb{Q}$ but not over finite fields $\mathbb{F}_q$ with $q \leq n-2$, and it has a nice Tutte polynomial. We then introduce a more general family of matroids, which we call *shifted matroids*. They are precisely the matroids whose independence complex is a shifted simplicial complex.

Chapter 2

Computing the Tutte polynomial of a hyperplane arrangement

2.1 Introduction

Much work has been devoted in recent years to studying hyperplane arrangements and, in particular, their characteristic polynomials. The polynomial $\chi_{\mathcal{A}}(q)$ is a very powerful invariant of the arrangement $\mathcal{A}$; it arises very naturally in many different contexts. There are many beautiful results about the characteristic polynomial of an arrangement. Two of them are the following.

Theorem 2.1.1 *(Zaslavsky, [74]) Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. The number of regions into which $\mathcal{A}$ dissects $\mathbb{R}^n$ is equal to $(-1)^n\chi_{\mathcal{A}}(-1)$. The number of regions which are relatively bounded is equal to $(-1)^n\chi_{\mathcal{A}}(1)$.*

Theorem 2.1.2 *(Orlik-Solomon, [43]) Let $\mathcal{A}$ be a central hyperplane arrangement in $\mathbb{C}^n$, and let $M_{\mathcal{A}} = \mathbb{C}^n - \bigcup_{H\in\mathcal{A}} H$ be the complement of the arrangement. Then*

$$\sum_{k\geq 0} \operatorname{rank} H^k(M_{\mathcal{A}}, \mathbb{Z})\, q^k = (-q)^n \chi_{\mathcal{A}}(-1/q).$$

Several authors have worked on computing the characteristic polynomials of specific hyperplane arrangements. This work has led to some very nice enumerative results; see for example [5], [51].

With the current state of affairs, it is surprising that nothing has been said about the Tutte polynomial of a hyperplane arrangement. When an arrangement $\mathcal{A}$ is central, there is a matroid $M_{\mathcal{A}}$ associated to it, and we can talk about the Tutte polynomial of that matroid. However, the Tutte polynomial has never even been defined for affine arrangements.

In Chapters 2 and 3 of this thesis, we will define and investigate the Tutte polynomial of a hyperplane arrangement. Chapter 2 is devoted to purely enumerative questions. We are particularly interested in computations for specific arrangements. Later, in Chapter 3, we will pursue matroid-theoretic questions. Central arrangements inherit Tutte polynomial properties from their associated matroids, and we will investigate to what extent such properties extend to affine arrangements.

In Section 2.2 we introduce the basic notions that we will need in this chapter. In Section 2.3 we define the Tutte polynomial of a hyperplane arrangement, and we present a finite field method for computing it. This is done in terms of the coboundary polynomial, a simple transformation of the Tutte polynomial. We derive some consequences of this method. Finally, in Section 2.4, we compute the Tutte polynomials of several families of arrangements. In particular, for deformations of the braid arrangement, we relate the computation of Tutte polynomials to some enumeration problems in classical combinatorics. As a consequence, we obtain several purely enumerative results about objects such as labeled trees, Dyck paths, alternating trees and semiorders.

2.2 Hyperplane arrangements

In this section we recall some of the basic concepts of hyperplane arrangements. For a more thorough introduction, we refer the reader to [44].

Given a field $\Bbbk$ and a positive integer n, an *affine hyperplane* in $\Bbbk^n$ is an $(n-1)$-dimensional affine subspace of $\Bbbk^n$. If we put a system of coordinates $x_1, \ldots, x_n$ on $\Bbbk^n$, a hyperplane can be seen as the set of points that satisfy a certain equation

$c_1x_1 + \cdots + c_nx_n = c$, where $c_1, \ldots, c_n, c \in \Bbbk$ and not all c_i's are equal to 0. A *hyperplane arrangement* $\mathcal{A}$ in $\Bbbk^n$ is a finite set of affine hyperplanes of $\Bbbk^n$. We will refer to hyperplane arrangements simply as *arrangements.* We will assume for simplicity that $\Bbbk = \mathbb{R}$ unless explicitly stated, although most of our results extend immediately to any field of characteristic zero.

We will say that an arrangement $\mathcal{A}$ is *central* if the hyperplanes in $\mathcal{A}$ have a non-empty intersection.[1] Similarly, we will say that a subset (or *subarrangement*) $\mathcal{B} \subseteq \mathcal{A}$ of hyperplanes is *central* if the hyperplanes in $\mathcal{B}$ have a non-empty intersection.

The *rank function* $r_{\mathcal{A}}$ is defined for each central subset $\mathcal{B}$ by the equation $r_{\mathcal{A}}(\mathcal{B}) = n - \dim \cap \mathcal{B}$. This function can be extended to a function $r_{\mathcal{A}} : 2^{\mathcal{A}} \to \mathbb{N}$, by defining the rank of a non-central subset $\mathcal{B}$ to be the largest rank of a central subset of $\mathcal{B}$. The *rank* of $\mathcal{A}$ is $r_{\mathcal{A}}(\mathcal{A})$, and it is denoted $r_{\mathcal{A}}$.

Alternatively, if the hyperplane H has defining equation $c_1x_1 + \cdots + c_nx_n = c$, assign to it the vector $v = (c_1, \ldots, c_n)$. Then define $r_{\mathcal{A}}(\{H_1, \ldots, H_k\})$ to be the dimension of the span of the corresponding vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$. It is easy to see that these two definitions of the rank function are equivalent. In particular, this means that the resulting function $r_{\mathcal{A}} : 2^{\mathcal{A}} \to \mathbb{N}$ is the rank function of a matroid. We will usually omit the subscripts when the underlying arrangement is clear, and simply write $r(\mathcal{B})$ and r for $r_{\mathcal{A}}(\mathcal{B})$ and $r_{\mathcal{A}}$, respectively.

The rank function gives us natural definitions of the usual concepts of matroid theory such as independent sets, bases, closed sets and circuits in the context of hyperplane arrangements. All of this will be done more naturally in the language of semimatroids in Chapter 3.

We will say that two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are *isomorphic*, and write $\mathcal{A}_1 \cong \mathcal{A}_2$, if there exists a bijection $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ which preserves the central subarrangements and the rank function.

To each hyperplane arrangement $\mathcal{A}$ we assign a partially ordered set, called the *intersection poset* of $\mathcal{A}$ and denoted $L_{\mathcal{A}}$. It consists of the non-empty intersections $H_{i_1} \cap \cdots \cap H_{i_k}$, ordered by reverse inclusion. This poset is graded, with rank function

[1] Sometimes we will call an arrangement *affine* to emphasize that it does not need to be central.

$r(H_{i_1} \cap \cdots \cap H_{i_k}) = r_{\mathcal{A}}(\{H_{i_1}, \ldots, H_{i_k}\})$, and a unique minimal element $\hat{0} = \mathbb{R}^n$.

The *characteristic polynomial* of $\mathcal{A}$ is

$$\chi_{\mathcal{A}}(q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) q^{n-r(x)}.$$

where μ denotes the Möbius function [56, Section 3.7] of $L_{\mathcal{A}}$.

Let $\mathcal{A}$ be an arrangement and let H be a hyperplane in $\mathcal{A}$. The arrangement $\mathcal{A} - \{H\}$ (or simply $\mathcal{A} - H$) is called the *deletion of* H *from* $\mathcal{A}$. It is an arrangement in $\mathbb{R}^n$. The arrangement $\mathcal{A}/H = \{H' \cap H \mid H' \in \mathcal{A} - H, H' \cap H \neq \emptyset\}$ is called the *contraction of* H *from* $\mathcal{A}$. It is an arrangement in H.

Notice, however, that some technical difficulties can arise. In a hyperplane arrangement $\mathcal{A}$, contracting a hyperplane H may give us repeated hyperplanes in the arrangement $\mathcal{A}/H$. If we are willing to allow repeated hyperplanes in arrangements then, when we contract one copy H of a repeated hyperplane, the other copy H' gives an element $H' \cap H = H$ in the "arrangement" $\mathcal{A}/H$, which is not a hyperplane.

The class of hyperplane arrangements, as we defined it, is not closed under deletion and contraction. This is problematic when we want to mirror matroid theoretic results in this context. There is an artificial solution to this problem: we can consider multisets $\{H_1, \ldots, H_k\}$ of subspaces of vector spaces V, where each H_i has dimension $\dim V - 1$ or $\dim V$. In other words, we allow repeated hyperplanes, and we allow the full space V to be regarded as a "hyperplane", mirroring a loop of a matroid. This class of objects *is* closed under deletion and contraction, but it is somewhat awkward to work with. A better solution is to think of arrangements as members of the class of semimatroids; a class that is also closed under deletion and contraction, and is more natural matroid-theoretically. We will adopt this point of view in Chapter 3.

2.3 Computing the Tutte polynomial

In [5], Athanasiadis introduced a powerful method for computing the characteristic polynomial of a subspace arrangement. He reduced the computation of characteristic polynomials to an enumeration in a vector space over a finite field. He used this method to compute explicitly the characteristic polynomial of several families of hyperplane arrangements, obtaining very nice enumerative results. As should be expected, this method only works when the equations defining the hyperplanes of the arrangement have integer (or rational) coefficients. Such an arrangement will be called a $\mathbb{Z}-$*arrangement*.

In [52], Reiner asked whether it is possible to use [52, Corollary 3] to compute explicitly the Tutte polynomials of some non-trivial families of representable matroids. Compared to all the work that has been done on computing characteristic polynomials explicitly, virtually nothing is known about computing Tutte polynomials.

In this section, we introduce a new method for computing Tutte polynomials of hyperplane arrangements. Our approach does not use Reiner's result; it is closer to Athanasiadis's method. In fact, Athanasiadis's result [5, Theorem 2.2] can be obtained as a special case of the main result of this section, Theorem 2.3.3, by setting $t = 0$.

After proving Theorem 2.3.3, we will present some of its consequences. We will then use it in Section 2.4 to compute explicitly the Tutte polynomials of several families of arrangements.

2.3.1 The Tutte and coboundary polynomials

Definition 2.3.1 *The* Tutte polynomial *of a hyperplane arrangement* $\mathcal{A}$ *is*

$$T_{\mathcal{A}}(q,t) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A} \\ \text{central}}} (q-1)^{r-r(\mathcal{B})}(t-1)^{|\mathcal{B}|-r(\mathcal{B})}, \qquad (2.3.1)$$

where the sum is over all central subsets $\mathcal{B} \subseteq \mathcal{A}$.

It will be useful for us to consider a simple transformation of the Tutte polynomial, first considered by Crapo [20] in the context of matroids.

Definition 2.3.2 *The* coboundary polynomial $\overline{\chi}_{\mathcal{A}}(q,t)$ *of an arrangement* $\mathcal{A}$ *is*

$$\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A} \\ \text{central}}} q^{r-r(\mathcal{B})}(t-1)^{|\mathcal{B}|}. \tag{2.3.2}$$

It is easy to check that

$$\overline{\chi}_{\mathcal{A}}(q,t) = (t-1)^r\, T_{\mathcal{A}}\left(\frac{q+t-1}{t-1}, t\right)$$

and

$$T_{\mathcal{A}}(x,y) = \frac{1}{(y-1)^r}\,\overline{\chi}_{\mathcal{A}}\left((x-1)(y-1), y\right).$$

Therefore, computing the coboundary polynomial of an arrangement is essentially equivalent to computing its Tutte polynomial. Our results can be presented more elegantly in terms of the coboundary polynomial.

2.3.2 The finite field method

Let $\mathcal{A}$ be a $\mathbb{Z}-$arrangement in $\mathbb{R}^n$, and let q be a prime power. The arrangement $\mathcal{A}$ induces an arrangement $\mathcal{A}_q$ in the vector space $\mathbb{F}_q^n$. If we consider the equations defining the hyperplanes of $\mathcal{A}$, and regard them as equations over $\mathbb{F}_q$, they define the hyperplanes of $\mathcal{A}_q$.

Say that $\mathcal{A}$ *reduces correctly* over $\mathbb{F}_q$ if the arrangements $\mathcal{A}$ and $\mathcal{A}_q$ are isomorphic. This does not always happen; sometimes the hyperplanes of $\mathcal{A}$ do not even become hyperplanes in $\mathcal{A}_q$. For example, the hyperplane $2x + 2y = 1$ in $\mathbb{R}^2$ becomes the empty "hyperplane" $0 = 1$ in $\mathbb{F}_2^2$. Sometimes independence is not preserved. For example, the independent hyperplanes $2x + y = 0$ and $y = 0$ in $\mathbb{R}^2$ become the same hyperplane in $\mathbb{F}_2^2$.

However, if q is a power of a large enough prime, $\mathcal{A}$ will reduce correctly over $\mathbb{F}_q$. To have $\mathcal{A} \cong \mathcal{A}_q$, we need central and independent subarrangements to be preserved. Cramer's rule lets us rephrase these conditions, in terms of certain determinants (formed by the coefficients of the hyperplanes in $\mathcal{A}$) being zero or non-zero. If we let

q be a power of a prime which is larger than all these determinants, we will guarantee that $\mathcal{A}$ reduces correctly over $\mathbb{F}_q$.

Theorem 2.3.3 *Let $\mathcal{A}$ be a $\mathbb{Z}-$arrangement in $\mathbb{R}^n$. Let q be a power of a large enough prime, and let $\mathcal{A}_q$ be the induced arrangement in $\mathbb{F}_q^n$. Then*

$$q^{n-r}\,\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{p\,\in\,\mathbb{F}_q^n} t^{h(p)} \tag{2.3.3}$$

where $h(p)$ denotes the number of hyperplanes of $\mathcal{A}_q$ that p lies on.

Proof. Let q be a power of a large enough prime, so that $\mathcal{A}$ reduces correctly over $\mathbb{F}_q$. For each $\mathcal{B} \subseteq \mathcal{A}$, let $\mathcal{B}_q$ be the subarrangement of $\mathcal{A}_q$ induced by it. For each $p \in \mathbb{F}_q^n$, let $H(p)$ be the set of hyperplanes of $\mathcal{A}_q$ that p lies on. From (2.3.2) we have

$$\begin{aligned}
q^{n-r}\,\overline{\chi}_{\mathcal{A}}(q,t) &= \sum_{\substack{\mathcal{B}\subseteq\mathcal{A}\\ \text{central}}} q^{n-r(\mathcal{B})}(t-1)^{|\mathcal{B}|}\\
&= \sum_{\substack{\mathcal{B}\subseteq\mathcal{A}\\ \text{central}}} q^{\dim\cap\mathcal{B}}(t-1)^{|\mathcal{B}|}\\
&= \sum_{\substack{\mathcal{B}_q\subseteq\mathcal{A}_q\\ \text{central}}} |\cap\mathcal{B}_q|\,(t-1)^{|\mathcal{B}_q|}\\
&= \sum_{\substack{\mathcal{B}_q\subseteq\mathcal{A}_q\\ \text{central}}} \sum_{p\,\in\,\cap\mathcal{B}_q} (t-1)^{|\mathcal{B}_q|}\\
&= \sum_{p\,\in\,\mathbb{F}_q^n} \sum_{\mathcal{B}_q\subseteq H(p)} (t-1)^{|\mathcal{B}_q|}\\
&= \sum_{p\,\in\,\mathbb{F}_q^n} (1+(t-1))^{h(p)},
\end{aligned}$$

as desired. □

In principle, Theorem 2.3.3 only computes $\overline{\chi}_{\mathcal{A}}(q,t)$ when q is a power of a large enough prime. In practice, however, when we compute the right-hand side of (2.3.3) for large prime powers q, we will get a polynomial function in q and t. Since the left-hand side is also a polynomial, these two polynomials must be equal.

Theorem 2.3.3 reduces the computation of coboundary polynomials (and hence

Tutte polynomials) to enumerating points in the finite vector space $\mathbb{F}_q^n$, according to a certain statistic. This method can be extremely useful when the hyperplanes of the arrangement are defined by simple equations. We will illustrate this in section 2.4.

We remark that Theorem 2.3.3 was also obtained independently by Welsh and Whittle [71].

2.3.3 Special cases and applications

Now we present some known results and some new results which follow from the finite field method. We start with two classical theorems which are special cases of Theorem 2.3.3.

Colorings of graphs

Given a graph G, Tutte [63] defined a matroid M_G called the *cycle matroid* of G; its Tutte polynomial is equal to the (graph-theoretic) Tutte polynomial of G.

From the point of view of arrangements, the construction is the following. Given a graph G on $[n]$, we associate to it an arrangement $\mathcal{A}_G$ in $\mathbb{R}^n$. It consists of the hyperplanes $x_i = x_j$, for all $1 \leq i < j \leq n$ such that ij is an edge in the graph G. Then we have that $T_G(q,t) = T_{\mathcal{A}_G}(q,t)$. We can define the coboundary polynomial for a graph like we did for arrangements, and then $\overline{\chi}_G(q,t) = \overline{\chi}_{\mathcal{A}_G}(q,t)$ also.

We shall now interpret Theorem 2.3.3 in this framework. It is easy to see that the rank of G is equal to $n - c$, where c is the number of connected components of G. Therefore the left-hand side of (2.3.3) is $q^c\,\overline{\chi}_G(q,t)$ in this case.

To interpret the right-hand side, notice that each point $p \in \mathbb{F}_q^n$ corresponds to a q-coloring of the vertices of G. The point $p = (p_1, \ldots, p_n)$ will correspond to the coloring κ_p of G which assigns color p_i to vertex i. A hyperplane $x_i = x_j$ contains p when $p_i = p_j$. This happens precisely when edge ij is *monochromatic* in κ_p; that is, when its two ends have the same color. We obtain the following known result.

Theorem 2.3.4 *([17, Proposition 6.3.26]) Let G be a graph with c connected com-*

ponents. Then

$$q^c\,\overline{\chi}_G(q,t) = \sum_{\substack{q-\text{colorings}\\ \kappa \text{ of } G}} t^{\text{mono}(\kappa)},$$

where mono(κ) *is the number of monochromatic edges in* κ*.*

Linear codes

Given positive integers $n \geq r$, an $[n,r]$ *linear code* C over $\mathbb{F}_q$ is an r-dimensional subspace of $\mathbb{F}_q^n$. A *generator matrix* for C is an $r \times n$ matrix U over $\mathbb{F}_q$, the rows of which form a basis for C. It is not difficult to see that the isomorphism class of the matroid on the columns of U depends only on C. We shall denote the corresponding matroid M_C.

The elements of C are called *codewords.* The *weight* $w(v)$ of a codeword is the cardinality of its support; that is, the number of non-zero coordinates of v. The *codeweight polynomial* of C is

$$A(C,q,t) = \sum_{v\,\in\, C} t^{w(v)}. \tag{2.3.4}$$

The translation of Theorem 2.3.3 to this setting is the following.

Theorem 2.3.5 *(Greene, [25]) For any linear code* C *over* $\mathbb{F}_q$*,*

$$A(C,q,t) = t^n\,\overline{\chi}_{M_C}\left(q, \frac{1}{t}\right).$$

Proof. Let $\mathcal{A}_C$ be the central arrangement corresponding to the columns of U. (We can call it $\mathcal{A}_C$ because, as stated above, its isomorphism class depends only on C.) This is a rank r arrangement in $\mathbb{F}_q^r$ such that $\overline{\chi}_{M_C}(q,\frac{1}{t}) = \overline{\chi}_{\mathcal{A}_C}(q,\frac{1}{t})$. Comparing (2.3.4) with Theorem 2.3.3, it remains to prove that

$$\sum_{v\,\in\, C} t^{w(v)} = \sum_{p\,\in\,\mathbb{F}_q^r} t^{n-h(p)}.$$

To do this, consider the bijection $\phi : \mathbb{F}_q^r \to C$ determined by right multiplication by U. If $u_1, \ldots, u_r$ are the row vectors of U, then ϕ sends $p = (p_1, \ldots, p_r) \in \mathbb{F}_q^r$ to the codeword $v_p = p_1 u_1 + \cdots + p_r u_r \in C$. For $1 \leq i \leq n$, p lies on the hyperplane determined by the $i - th$ column of U if and only if the i-th coordinate of v_p is equal to zero. Therefore $h(p) = n - w(v_p)$. This completes the proof. □

Deletion-contraction

The point of view of Theorem 2.3.3 can be used to give a nice enumerative proof of the deletion-contraction formula for the Tutte polynomial of an arrangement. Once again, this formula is better understood in the context of semimatroids, as we will do in Proposition 3.8.3. For now, leaving matroid-theoretical issues aside, we only wish to present a special case of it as a nice application.

Proposition 2.3.6 *Let $\mathcal{A}$ be a hyperplane arrangement, and let H be a hyperplane in $\mathcal{A}$ such that $r_{\mathcal{A}}(\mathcal{A} - H) = r_{\mathcal{A}}$. Then $T_{\mathcal{A}}(q,t) = T_{\mathcal{A}-H}(q,t) + T_{\mathcal{A}/H}(q,t)$.*

Proof. Because there will be several arrangements involved, let $h(\mathcal{B}, p)$ denote the number of hyperplanes in $\mathcal{B}_q$ that p lies on. Then

$$\begin{aligned}
q^{n-r}\, \overline{\chi}_{\mathcal{A}}(q,t) &= \sum_{p \in \mathbb{F}_q^n} t^{h(\mathcal{A},p)} \\
&= \sum_{p \in \mathbb{F}_q^n - H} t^{h(\mathcal{A},p)} + \sum_{p \in H} t^{h(\mathcal{A},p)} \\
&= \sum_{p \in \mathbb{F}_q^n - H} t^{h(\mathcal{A}-H,p)} + \sum_{p \in H} t^{h(\mathcal{A}-H,p)+1} \\
&= \sum_{p \in \mathbb{F}_q^n} t^{h(\mathcal{A}-H,p)} + (t-1) \sum_{p \in H} t^{h(\mathcal{A}-H,p)} \\
&= q^{n-r}\, \overline{\chi}_{\mathcal{A}-H}(q,t) + (t-1) q^{(n-1)-(r-1)}\, \overline{\chi}_{\mathcal{A}/H}(q,t).
\end{aligned}$$

We conclude that $\overline{\chi}_{\mathcal{A}}(q,t) = \overline{\chi}_{\mathcal{A}-H}(q,t) + (t-1)\, \overline{\chi}_{\mathcal{A}/H}(q,t)$, which is equivalent to the deletion-contraction formula for Tutte polynomials. □

A probabilistic interpretation

Theorem 2.3.7 *Let $\mathcal{A}$ be an arrangement and let $0 \leq t \leq 1$ be a real number. Let $\mathcal{B}$ be a random subarrangement of $\mathcal{A}$, obtained by independently removing each hyperplane from $\mathcal{A}$ with probability t. Then the expected characteristic polynomial $\chi_{\mathcal{B}}(q)$ of $\mathcal{B}$ is $q^{n-r}\overline{\chi}_{\mathcal{A}}(q,t)$.*

Proof. We have

$$\begin{aligned} E[\chi_{\mathcal{B}}(q)] &= \sum_{\mathcal{C}\subseteq\mathcal{A}} P[\mathcal{B}=\mathcal{C}]\,\chi_{\mathcal{C}}(q) \\ &= \sum_{\mathcal{C}\subseteq\mathcal{A}} P[\mathcal{B}=\mathcal{C}]\,|\,\mathbb{F}_q^n - \cup\mathcal{C}_q| \\ &= \sum_{p\,\in\,\mathbb{F}_q^n}\ \sum_{\substack{\mathcal{C}\subseteq\mathcal{A}\\ p\notin\cup\mathcal{C}_q}} P[\mathcal{B}=\mathcal{C}], \end{aligned}$$

where in the second step we have used Athanasiadis's result [5]; that is, the case $t=0$ of Theorem 2.3.3.

Recall that $H(p)$ denotes the set of hyperplanes in $\mathcal{A}_q$ containing p. Then

$$\begin{aligned} E[\chi_{\mathcal{B}}(q)] &= \sum_{p\,\in\,\mathbb{F}_q^n} P[\mathcal{B}_q\cap H(p)=\emptyset] \\ &= \sum_{p\,\in\,\mathbb{F}_q^n} t^{h(p)}, \end{aligned}$$

which is precisely what we wanted to show. □

A Möbius formula

Theorem 2.3.8 *For an arrangement $\mathcal{A}$ and an affine subspace x in the intersection poset $L_{\mathcal{A}}$, let $h(x)$ be the number of hyperplanes of $\mathcal{A}$ containing x. Then*

$$\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{x\leq y \text{ in } L_{\mathcal{A}}} \mu(x,y)\, q^{r-r(y)} t^{h(x)}.$$

Proof. Consider the arrangement $\mathcal{A}$ restricted to $\mathbb{F}_q^n$, where q is a power of a large enough prime, so that $\mathcal{A}$ reduces correctly over $\mathbb{F}_q$. Given $x \in L_{\mathcal{A}_q}$, let $P(x)$ be the set of points in $\mathbb{F}_q^n$ which are contained in x, and are not contained in any y such that $y > x$ in $L_{\mathcal{A}_q}$. Then the set x is partitioned by the sets $P(y)$ for $y \geq x$, so we have

$$q^{\dim x} = |x| = \sum_{y \geq x} |P(y)|.$$

By the Möbius inversion formula [56, Proposition 3.7.1] we have

$$|P(x)| = \sum_{y \geq x} \mu(x, y)\, q^{\dim y}.$$

Now, from Theorem 2.3.3 we know that

$$\begin{aligned} q^{n-r}\overline{\chi}_{\mathcal{A}}(q,t) &= \sum_{x \in L_{\mathcal{A}}} \sum_{p \in P(x)} t^{h(p)} = \sum_{x \in L_{\mathcal{A}}} |P(x)|\, t^{h(x)} \\ &= \sum_{x \leq y \text{ in } L_{\mathcal{A}}} \mu(x, y)\, q^{n-r(y)} t^{h(x)}, \end{aligned}$$

as desired. □

2.4 The finite field method in practice

In this section we use Theorem 2.3.3 to compute the coboundary polynomials of several families of arrangements. As remarked at the beginning of Section 2.3.1, this is essentially the same as computing their Tutte polynomials.

2.4.1 Coxeter arrangements

To illustrate how our finite field method works, we start by presenting some simple examples.

Let Φ be an irreducible crystallographic root system in $\mathbb{R}^n$, with the standard inner product, and let W be its associated Weyl group. The Coxeter arrangement of type W consists of the hyperplanes $(\alpha, x) = 0$ for each $\alpha \in \Phi^+$. See [31] for an introduction to root systems and Weyl groups, and [44, Chapter 6] or [11, Section 2.3] for more information on Coxeter arrangements.

In this section we compute the coboundary polynomials of the Coxeter arrangements of type A_n, B_n and D_n. (The arrangement of type C_n is the same as the arrangement of type B_n.) The best way to state our results is to compute the exponential generating function for the coboundary polynomials of each family.

The following three theorems have never been stated explicitly in the literature in this form. Theorem 2.4.1 is equivalent to a result of Tutte [62], who computed the Tutte polynomial of the complete graph. It is also an immediate consequence of a more general theorem of Stanley [59, (15)]. Theorems 2.4.2 and 2.4.3 are implicit in the work of Zaslavsky [75].

Theorem 2.4.1 *Let $\mathcal{A}_n$ be the Coxeter arrangement of type A_{n-1} in $\mathbb{R}^n$, consisting of the hyperplanes $x_i = x_j$ for $1 \leq i < j \leq n$.*[2] *We have*

$$1 + q \sum_{n \geq 1} \overline{\chi}_{\mathcal{A}_n}(q,t) \frac{x^n}{n!} = \left(\sum_{n \geq 0} t^{\binom{n}{2}} \frac{x^n}{n!} \right)^q .$$

Proof. For $n \geq 1$ we have that

$$q \, \overline{\chi}_{\mathcal{A}_n}(q,t) = \sum_{p \, \in \, \mathbb{F}_q^n} t^{h(p)}.$$

for all powers of a large enough prime q, according to Theorem 2.3.3. For each $p \in \mathbb{F}_q^n$, if we let $A_k = \{ i \in [n] \, | \, p_i = k \}$ for $0 \leq k \leq q-1$, then $h(p) = \binom{|A_0|}{2} + \cdots + \binom{|A_{q-1}|}{2}$. Thus

$$q \, \overline{\chi}_{\mathcal{A}_n}(q,t) = \sum_{A_0 \cup \cdots \cup A_{q-1} = [n]} t^{\binom{|A_0|}{2} + \cdots + \binom{|A_{q-1}|}{2}}$$

where the sum is over all weak ordered q-partitions of $[n]$. The compositional formula for exponential generating functions [60, Proposition 5.1.3], [8] implies the desired result. □

[2]This arrangement is also known as the *braid arrangement.*

Theorem 2.4.2 *Let $\mathcal{B}_n$ be the Coxeter arrangement of type B_n in $\mathbb{R}^n$, consisting of the hyperplanes $x_i = x_j$ and $x_i + x_j = 0$ for $1 \le i < j \le n$, and the hyperplanes $x_i = 0$ for $1 \le i \le n$. We have*

$$\sum_{n\ge 0} \overline{\chi}_{\mathcal{B}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\ge 0} 2^n t^{\binom{n}{2}} \frac{x^n}{n!}\right)^{\frac{q-1}{2}} \left(\sum_{n\ge 0} t^{n^2}\frac{x^n}{n!}\right).$$

Proof. Let q be a power of a large enough prime, and let $s = \frac{q-1}{2}$. Now for each $p \in \mathbb{F}_q^n$, if we let $B_k = \{i \in [n] \mid p_i = k \text{ or } p_i = q-k\}$ for $0 \le k \le s$, we have that $h(p) = |B_0|^2 + \binom{|B_1|}{2} + \cdots + \binom{|B_s|}{2}$. Also, given a weak ordered partition $(B_0, \ldots, B_s)$ of $[n]$, there are $2^{|B_1|+\cdots+|B_s|}$ points of p which correspond to it: for each $i \in B_k$ with $k \ne 0$, we get to choose whether p_i is equal to k or to $q-k$. Therefore

$$q\,\overline{\chi}_{\mathcal{B}_n}(q,t) = \sum_{B_0\cup\cdots\cup B_s=[n]} t^{|B_0|^2}\left(2^{|B_1|}t^{\binom{|B_1|}{2}}\right)\cdots\left(2^{|B_s|}t^{\binom{|B_s|}{2}}\right),$$

and the compositional formula for exponential generating functions implies Theorem 2.4.2. □

Theorem 2.4.3 *Let $\mathcal{D}_n$ be the Coxeter arrangement of type D_n in $\mathbb{R}^n$, consisting of the hyperplanes $x_i = x_j$ and $x_i + x_j = 0$ for $1 \le i < j \le n$. We have*

$$\sum_{n\ge 0} \overline{\chi}_{\mathcal{D}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\ge 0} 2^n t^{\binom{n}{2}} \frac{x^n}{n!}\right)^{\frac{q-1}{2}} \left(\sum_{n\ge 0} t^{n(n-1)}\frac{x^n}{n!}\right).$$

We omit the details of the proof of Theorem 2.4.3, which is essentially the same as the proof of Theorem 2.4.2.

Setting $t = 0$ in Theorems 2.4.1, 2.4.2 and 2.4.3, it is easy to recover the well-known formulas for the characteristic polynomials of the above arrangements:

$$\begin{aligned}
\chi_{\mathcal{A}_n}(q) &= q(q-1)(q-2)\cdots(q-n+1),\\
\chi_{\mathcal{B}_n}(q) &= (q-1)(q-3)\cdots(q-2n+1),\\
\chi_{\mathcal{D}_n}(q) &= (q-1)(q-3)\cdots(q-2n+3)(q-n+1).
\end{aligned}$$

2.4.2 Two more examples

Theorem 2.4.4 *Let $\mathcal{A}_n^\#$ be a generic deformation of the arrangement $\mathcal{A}_n$, consisting of the hyperplanes $x_i - x_j = a_{ij}$ $(1 \le i < j \le n)$, where the a_{ij} are generic real numbers* [3]*. For $n \ge 1$,*

$$q\,\overline{\chi}_{\mathcal{A}_n^\#}(q,t) = \sum_F q^{n-e(F)}(t-1)^{e(F)}$$

where the sum is over all forests F on $[n]$, and $e(F)$ denotes the number of edges of F. Also,

$$1 + q\sum_{n\ge 1}\overline{\chi}_{\mathcal{A}_n^\#}(q,t)\frac{x^n}{n!} = \left(\sum_{n\ge 0} f(n)\frac{x^n(t-1)^n}{n!}\right)^{\frac{q}{t-1}},$$

where $f(n)$ is the number of forests on $[n]$.

Proof. It is possible to prove Theorem 2.4.4 using our finite field method, as we did in the previous section. However, it will be easier to proceed directly from (2.3.2), the definition of the coboundary polynomial.

To each subarrangement $\mathcal{B}$ of $\mathcal{A}_n^\#$ we can assign a graph $G_\mathcal{B}$ on the vertex set $[n]$, by letting edge ij be in $G_\mathcal{B}$ if and only if the hyperplane $x_i - x_j = a_{ij}$ is in $\mathcal{B}$. Since the a_{ij}'s are generic, the subarrangement $\mathcal{B}$ is central if and only if the corresponding graph $G_\mathcal{B}$ is a forest. For such a $\mathcal{B}$, it is clear that $|\mathcal{B}| = r(\mathcal{B}) = e(G_\mathcal{B})$. Hence,

$$\begin{aligned}\overline{\chi}_{\mathcal{A}_n^\#}(q,t) &= \sum_{\substack{\mathcal{B}\subseteq\mathcal{A}_n^\# \\ \text{central}}} q^{r-r(\mathcal{B})}(t-1)^{|\mathcal{B}|} \\ &= \sum_F q^{(n-1)-e(F)}(t-1)^{e(F)},\end{aligned}$$

proving the first claim. Now let $c(F) = n - e(F)$ be the number of connected com-

[3] The a_{ij} are "generic" if no n of the hyperplanes have a non-empty intersection, and any non-empty intersection of k hyperplanes has rank k. This can be achieved, for example, by requiring that the a_{ij}'s are linearly independent over the rational numbers. Almost all choices of the a_{ij}'s are generic.

ponents of F. We have

$$\begin{aligned} 1+q\sum_{n\geq 1}\overline{\chi}_{\mathcal{A}_n^{\#}}(q,t)\frac{x^n}{n!} &= \sum_{n\geq 0}\sum_{F \text{ on } [n]}\left(\frac{q}{t-1}\right)^{c(F)}\frac{x^n(t-1)^n}{n!} \\ &= \left(\sum_{n\geq 0} f(n)\frac{x^n(t-1)^n}{n!}\right)^{\frac{q}{t-1}} \end{aligned}$$

by the compositional formula for exponential generating functions. □

Theorem 2.4.5 *The threshold arrangement $\mathcal{T}_n$ in $\mathbb{R}^n$ consists of the hyperplanes $x_i + x_j = 0$, for $1 \leq i < j \leq n$. For all $n \geq 0$ we have*

$$\overline{\chi}_{\mathcal{T}_n}(q,t) = \sum_G q^{bc(G)}(t-1)^{e(G)},$$

where the sum is over all graphs G on $[n]$. Here $bc(G)$ is the number of connected components of G which are bipartite, and $e(G)$ is the number of edges of G. Also,

$$\sum_{n\geq 0}\overline{\chi}_{\mathcal{T}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}t^{k(n-k)}\frac{x^n}{n!}\right)^{\frac{q-1}{2}}\left(\sum_{n\geq 0}t^{\binom{n}{2}}\frac{x^n}{n!}\right).$$

Proof. Once again, the proof of the first claim is easier using the definition of the coboundary polynomial. Every subarrangement $\mathcal{B}$ of $\mathcal{T}_n$ is central, and we can assign to it a graph $G_{\mathcal{B}}$ as in the proof of Theorem 2.4.4. In view of (2.3.2), we only need to check that $r(\mathcal{B}) = n - bc(G_{\mathcal{B}})$ and $|\mathcal{B}| = e(G_{\mathcal{B}})$. The second claim is trivial. To prove the first one, we show that $\dim(\cap\mathcal{B}) = bc(G_{\mathcal{B}})$.

Consider a point p in $\cap\mathcal{B}$. We know that, if ab is an edge in $G_{\mathcal{B}}$, then $p_a = -p_b$. If vertex i is in a connected component C of $G_{\mathcal{B}}$, then the value of p_i determines the value of p_j for all j in C: $p_j = p_i$ if the distance between i and j is even, and $p_j = -p_i$ if the distance between i and j is odd. If C is bipartite, then this determines the values of the p_j's consistently. If C is not bipartite, take a cycle of odd length and a k in it. We get that $p_k = -p_k$, so $p_k = 0$; therefore we must have $p_j = 0$ for all $j \in C$.

Therefore, to specify a point p in $\cap\mathcal{B}$, we split $G_{\mathcal{B}}$ into its connected components.

We know that $p_i = 0$ for all i in connected components which are not bipartite. To determine the remaining coordinates of p we have to specify the value of p_j for exactly one j in each bipartite connected component. Therefore $\dim(\cap\mathcal{B}) = bc(G_{\mathcal{B}})$, as desired.

From this point, it is possible to prove the second claim of Theorem 2.4.5 using the compositional formula for exponential generating functions, in the same way that we proved Theorem 2.4.4. However, the work involved is considerable, and it is much simpler to use our finite field method, Theorem 2.3.3, in this case. The proof that we obtain is very similar to the proofs of Theorems 2.4.1, 2.4.2 and 2.4.3, so we omit the details. □

2.4.3 Deformations of the braid arrangement

A *deformation of the braid arrangement* is an arrangement in $\mathbb{R}^n$ consisting of the hyperplanes $x_i - x_j = a_{ij}^{(1)}, \ldots, a_{ij}^{(k_{ij})}$ for $1 \leq i < j \leq n$, where the k_{ij} are non-negative integers, and the $a_{ij}^{(r)}$ are real numbers. Such arrangements have been studied extensively by Athanasiadis [7] and Postnikov and Stanley [51]. In this section we study their coboundary polynomials.

Theorem 2.4.6 *Let $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \ldots)$ be a sequence of arrangements satisfying the following properties:* [4]

1. *$\mathcal{E}_n$ is a deformation of the braid arrangement in $\mathbb{R}^n$.*

2. *$r(\mathcal{E}_n) = n - 1$.*

3. *For any subset S of $[n]$, the subarrangement $\mathcal{E}_n^S \subseteq \mathcal{E}_n$, which consists of the hyperplanes in $\mathcal{E}_n$ of the form $x_i - x_j = c$ with $i, j \in S$, is isomorphic to the arrangement $\mathcal{E}_{|S|}$.*

Then

$$1 + q \sum_{n \geq 1} \overline{\chi}_{\mathcal{E}_n}(q,t) \frac{x^n}{n!} = \left(\sum_{n \geq 0} \overline{\chi}_{\mathcal{E}_n}(1,t) \frac{x^n}{n!} \right)^q .$$

[4] Such a sequence is called an *exponential sequence of arrangements.*

The special case $t = 0$ of this result is due to Stanley [57, Theorem 1.2]. Theorem 2.4.6 is an easy consequence of the upcoming Proposition 2.4.10 and the compositional formula for exponential generating functions.

The most natural examples of exponential sequences of arrangements are the following. Fix a set A of k distinct integers $a_1 < \ldots < a_k$. Let $\mathcal{E}_n$ be the arrangement in $\mathbb{R}^n$ consisting of the hyperplanes

$$x_i - x_j = a_1, \ldots, a_k \qquad 1 \leq i < j \leq n. \tag{2.4.1}$$

Then $(\mathcal{E}_0, \mathcal{E}_1, \ldots)$ is an exponential sequence of arrangements and Theorem 2.4.6 applies to this case. In fact, we can say much more about this type of arrangement.

After proving the results in this section, we found out that Postnikov and Stanley [51] had used similar techniques in computing the characteristic polynomials of these types of arrangements. Therefore, for consistency, we will use the terminology that they introduced.

Definition 2.4.7 *A* graded graph *is a triple $G = (V_G, E_G, h_G)$, where V_G is a linearly ordered set of vertices (usually $V_G = [n]$), E_G is a set of (non-oriented) edges, and h_G is a function $h_G : V \to \mathbb{N}$, called a* grading.

We will drop the subscripts when the underlying graded graph is clear. We will refer to $h(v)$ as the *height* of vertex v. The *height* of G, denoted $h(G)$, is the largest height of a vertex of G.

Definition 2.4.8 *Let G be a graded graph and r be a non-negative integer. The r-th* level *of G is the set of vertices v such that $h(v) = r$. G is* planted *if each one of its connected components has a vertex on the 0-th level.*

Definition 2.4.9 *If $u < v$ are connected by edge e in a graded graph G, the* slope *of e is $s(e) = h(u) - h(v)$. G is an A-*graph *if the slopes of all edges of G are in $A = \{a_1, \ldots, a_k\}$.*

Recall that, for a graph G, we let $e(G)$ be the number of edges and $c(G)$ be the number of connected components of G. We also let $v(G)$ be the number of vertices of G.

Proposition 2.4.10 *Let $\mathcal{E}_n$ be the arrangement (2.4.1). Then, for $n \geq 1$,*

$$q\,\overline{\chi}_{\mathcal{E}_n}(q,t) = \sum_G q^{c(G)}(t-1)^{e(G)},$$

where the sum is over all planted graded A-graphs on $[n]$.

Proof. We associate to each planted graded A-graph $G = (V, E, h)$ on $[n]$ a central subarrangement $\mathcal{A}_G$ of $\mathcal{E}_n$. It consists of the hyperplanes $x_i - x_j = h(i) - h(j)$, for each $i < j$ such that ij is an edge in G. This is a subarrangement of $\mathcal{E}_n$ because $h(i) - h(j)$, the slope of edge ij, is in A. It is central because the point $(h(1), \ldots, h(n)) \in \mathbb{R}^n$ belongs to all these hyperplanes.

This is in fact a bijection between planted graded A-graphs on $[n]$ and central subarrangements of $\mathcal{E}_n$. To see this, take a central subarrangement $\mathcal{A}$. We will recover the planted graded A-graph G that it came from. For each pair (i, j) with $1 \leq i < j \leq n$, $\mathcal{A}$ can have at most one hyperplane of the form $x_i - x_j = a_t$. If this hyperplane is in $\mathcal{A}$, we must put edge ij in G, and demand that the heights $h(i)$ and $h(j)$ satisfy $h(i) - h(j) = a_t$. When we do this for all the hyperplanes in $\mathcal{A}$, the height requirements that we introduce are consistent, because $\mathcal{A}$ is central. However, these requirements do not fully determine the heights of the vertices; they only determine the relative heights within each connected component of G. Since we want G to be planted, we demand that the vertices with the lowest height in each connected component of G should have height 0. This does determine G completely, and clearly $\mathcal{A} = \mathcal{A}_G$.

Example. Consider an arrangement $\mathcal{E}_8$ in $\mathbb{R}^8$, with a subarrangement consisting of the hyperplanes $x_1 - x_2 = 4, x_1 - x_3 = -1, x_1 - x_6 = 0, x_1 - x_8 = 1, x_2 - x_3 = -5$ and $x_4 - x_7 = 2$. Figure 2-1 shows the planted graded A-graph corresponding to this subarrangement.

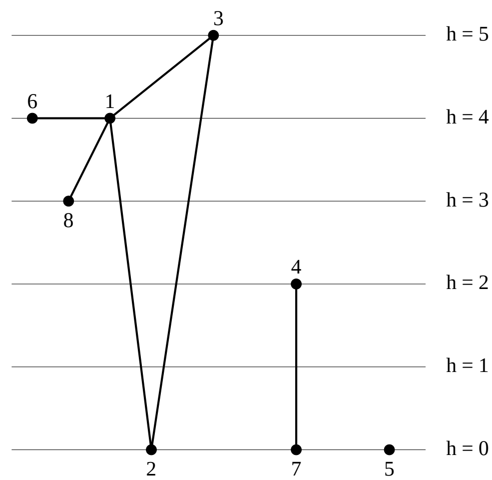

Figure 2-1: The planted graded A-graph corresponding to a subarrangement of $\mathcal{E}_8$.

With this bijection in hand, and keeping (2.3.2) in mind, it remains to show that $r(\mathcal{A}_G) = n - c(G)$ and $|\mathcal{A}_G| = e(G)$. The second of these claims is trivial. We omit the proof of the first one which is very similar to, and simpler than, that of $r(\mathcal{B}) = n - bc(G_{\mathcal{B}})$ in our proof of Theorem 2.4.5. □

Theorem 2.4.11 *Let $\mathcal{E}_n$ be the arrangement (2.4.1), and let*

$$A_r(t,x) = \sum_{n\geq 0} \left(\sum_{f:[n]\to[r]} t^{a(f)} \right) \frac{x^n}{n!}, \tag{2.4.2}$$

where $a(f)$ denotes the number of pairs (i,j) with $1 \leq i < j \leq n$ such that $f(i) - f(j) \in A$. Then

$$1 + q \sum_{n\geq 1} \overline{\chi}_{\mathcal{E}_n}(q,t) \frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(t,x)}{A_{r-1}(t,x)} \right)^q. \tag{2.4.3}$$

Remark. The limit in (2.4.3) is a limit in the sense of convergence of formal power series. Let $F_1(t,x), F_2(t,x), \ldots$ be a sequence of formal power series. We say that $\lim_{n\to\infty} F_n(t,x) = F(t,x)$ if, for all a and b, the coefficient of $t^a x^b$ in $F_n(t,x)$ is equal to the coefficient of $t^a x^b$ in $F(t,x)$ for all n larger than some constant $N(a,b)$. For more information on this notion of convergence, see [56, Section 1.1] or [42].

Proof of Theorem 2.4.11. First we prove that

$$A_r(t,x) = \sum_G (t-1)^{e(G)} \frac{x^{v(G)}}{v(G)!} \tag{2.4.4}$$

where the sum is over all graded A-graphs G of height less than r. The coefficient of $\frac{x^n}{n!}$ in the right-hand side of (2.4.4) is $\sum_G (t-1)^{e(G)}$, summing over all graded A-graphs G on $[n]$ with height less than r. We have

$$\begin{aligned}
\sum_G (t-1)^{e(G)} &= \sum_{h:[n]\to[0,r-1]} \sum_{\substack{G \text{ such that} \\ h_G = h}} (t-1)^{e(G)} \\
&= \sum_{h:[n]\to[0,r-1]} (1+(t-1))^{a(h)} \\
&= \sum_{f:[n]\to[r]} t^{a(f)}
\end{aligned}$$

The only tricky step here is the second: if we want all graded A-graphs G on $[n]$ with a specified grading h, we need to consider the possible choices of edges of the graph. Any edge ij can belong to the graph, as long as $h(i) - h(j) \in A$, so there are $a(h)$ possible edges.

Equation (2.4.4) suggests the following definitions. Let

$$B_r(t,x) = \sum_G t^{e(G)} \frac{x^{v(G)}}{v(G)!}$$

where the sum is over all *planted* graded A-graphs G of height less than r, and let

$$B(t,x) = \sum_G t^{e(G)} \frac{x^{v(G)}}{v(G)!}$$

where the sum is over all *planted* graded A-graphs G.

The equation

$$1 + q \sum_{n\geq 1} \overline{\chi}_{\mathcal{E}_n}(q,t) \frac{x^n}{n!} = B(t-1,x)^q, \tag{2.4.5}$$

follows from Proposition 2.4.10, using either Theorem 2.4.6 or the compositional formula for exponential generating functions.

Now we claim that $B(t,x) = \lim_{r\to\infty} B_r(t,x)$. Notice that, in a planted graded A-graph G with e edges and v vertices, each vertex has a path of length at most v which connects it to a vertex on the 0-th level. Recalling that $a_1 < \ldots < a_k$ we see that $h(G) \leq v \cdot \max(-a_1, a_k)$, so the coefficients of $t^e \frac{x^v}{v!}$ in $B_r(t,x)$ and $B(t,x)$ are equal for $r > v \cdot \max(-a_1, a_k)$.

With a little bit of care, it then follows easily that

$$B(t-1,x) = \lim_{r\to\infty} B_r(t-1,x). \tag{2.4.6}$$

Here it is necessary to check that $B(t-1,x)$ is, indeed, a formal power series. This follows from the observation that the coefficient of $\frac{x^n}{n!}$ in $B(t,x)$ is a polynomial in t of degree at most $\binom{n}{2}$. We know that for some formal power series $f(t)$ (like e^t, for example), $f(t-1)$ is not a well-defined formal power series. In our case, however, this is not a problem and (2.4.6) is valid. Once again, see [56, Section 1.1] for more information on these technical details.

Next, we show that

$$B_r(t-1,x) = A_r(t,x)/A_{r-1}(t,x) \tag{2.4.7}$$

or, equivalently, that $A_r(t,x) = B_r(t-1,x)A_{r-1}(t,x)$. The multiplication formula for exponential generating functions ([60, Proposition 5.1.1]) and (2.4.4) give us a combinatorial interpretation of this identity. We need to show that the ways of putting the structure of a graded A-graph G with $h(G) < r$ on $[n]$ can be put in correspondence with the ways of doing the following: first splitting $[n]$ into two disjoint sets S_1 and S_2, then putting the structure of a *planted* graded A-graph G_1 with $h(G_1) < r$ on S_1, and then putting the structure of a graded A-graph G_2 with $h(G_2) < r-1$ on S_2. We also need that, in that correspondence, $(t-1)^{e(G)} = (t-1)^{e(G_1)}(t-1)^{e(G_2)}$.

We do this as follows. Let G be a graded A-graph G with $h(G) < r$. Let G_1 be the union of the connected components of G which contain a vertex on the 0-th level. Put a grading on G_1 by defining $h_{G_1}(v) = h_G(v)$ for $v \in G_1$. Let $G_2 = G - G_1$. It is

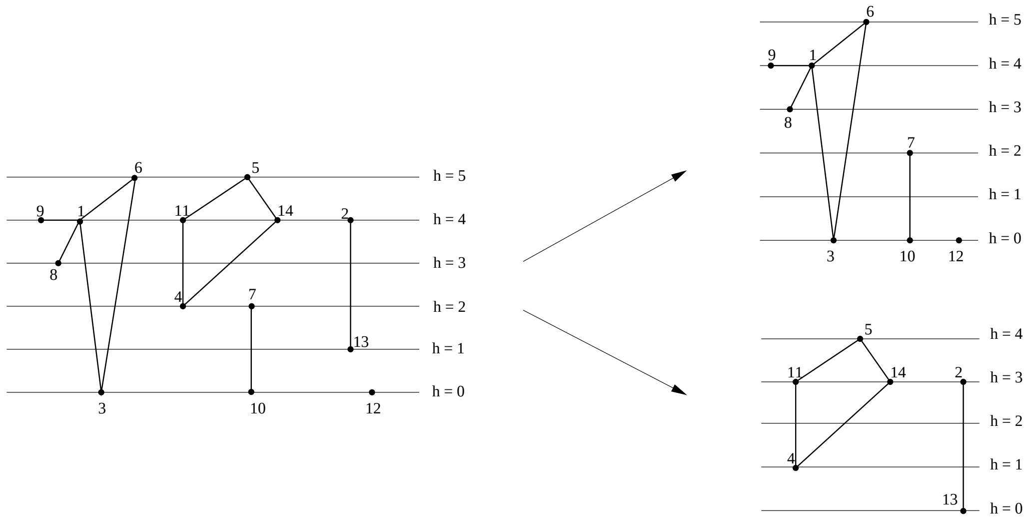

Figure 2-2: The decomposition of a graded A-graph.

clear that $h_G(v) \geq 1$ for all $v \in G_2$; therefore we can put a grading on G_2 by defining $h_{G_2}(v) = h_G(v) - 1$ for $v \in G_2$. G_1 is a planted graded A-graph with $h(G_1) < r$, and G_2 is a graded A-graph with $h(G_2) < r - 1$. Figure 2-2 illustrates this decomposition with an example.

Our map from G to a pair (G_1, G_2) is a one-to-one correspondence. Any pair (G_1, G_2), with G_1 planted of height less than r and G_2 of height less than $r - 1$, arises from a decomposition of some G of height less than r in this way. It is clear how to recover G from G_1 and G_2. Also, it is clear from the construction of the correspondence that $(t-1)^{e(G)} = (t-1)^{e(G_1)}(t-1)^{e(G_2)}$. This completes the proof of (2.4.7).

Now we just have to put together (2.4.5), (2.4.6) and (2.4.7) to complete the proof of Theorem 2.4.11. □

The *Catalan arrangement* C_n in $\mathbb{R}^n$ consists of the hyperplanes

$$x_i - x_j = -1, 0, 1 \qquad 1 \leq i < j \leq n.$$

When the arrangement in Theorem 2.4.11 is a subarrangement of the Catalan arrangement, we can say more about the power series A_r of (2.4.2). Let

$$A(t,x,y) = \sum_r A_r(t,x)y^r = \sum_{n\geq 0}\sum_{r\geq 0}\left(\sum_{f:[n]\to[r]} t^{a(f)}\right)\frac{x^n}{n!}y^r$$

and let

$$S(t,x,y) = \sum_{n\geq 0}\sum_{r\geq 0}\left(\sum_{f:[n]\to[r]} t^{a(f)}\right)\frac{x^n}{n!}y^r \tag{2.4.8}$$

where the inner sum is over all *surjective* functions $f : [n] \to [r]$. The following proposition reduces the computation of $A(t,x,y)$ to the computation of $S(t,x,y)$, which is often easier in practice.

Proposition 2.4.12 *If $A \subseteq \{-1,0,1\}$ in the notation of Theorem 2.4.11, we have*

$$A(t,x,y) = \frac{S(t,x,y)}{1 - yS(t,x,y)}$$

Proof. Once again, we think of this as an identity about exponential generating functions in the variable x. Fix n, r, and $f : [n] \to [r]$. Let the image of f be $\{1,\ldots,m_1-1\} \cup \{m_1+1,\ldots,m_1+m_2-1\} \cup \cdots \cup \{m_1+\cdots+m_{k-1}+1,\ldots,m_1+\cdots+m_k-1\} = M_1 \cup \cdots \cup M_k$, so that $[r] - \operatorname{Im} f = \{m_1, m_1+m_2, \ldots, m_1+\cdots+m_{k-1}\}$. Here $m_1,\ldots,m_k$ are arbitrary positive integers such that $m_1+\cdots+m_k-1 = r$. For $1 \leq i \leq k$, let f_i be the restriction of f to $f^{-1}(M_i)$; it maps $f^{-1}(M_i)$ surjectively to M_i. Then we can "decompose" f in a unique way into the k *surjective* functions $f_1,\ldots,f_k$. The weight $w(f)$ corresponding to f in $A(t,x,y)$ is $t^{a(f)}y^r$, while the weight $w(f_i)$ corresponding to f_i in $S(t,x,y)$ is $t^{a(f_i)}y^{m_i-1}$.

Now observe that $a(f) = a(f_1)+\cdots+a(f_k)$: whenever we have a pair of numbers $1 \leq i < j \leq n$ counted by $a(f)$, since $f(i) - f(j) \in \{-1,0,1\}$, we know that $f(i)$ and $f(j)$ must be in the same M_h. Therefore i and j are in the same $f^{-1}(M_h)$, and this pair is also counted in $a(f_h)$. We also have that $r = (m_1-1)+\cdots+(m_k-1)+(k-1)$. Therefore $w(f) = w(f_1)\cdots w(f_k)y^{k-1}$. It follows from the compositional formula for exponential generating functions that

$$\begin{aligned} A(t,x,y) &= \sum_{k\geq 1} S(t,x,y)^k y^{k-1} \\ &= \frac{S(t,x,y)}{1-yS(t,x,y)} \end{aligned}$$

as desired. □

Considering the different subsets of $\{-1,0,1\}$, we get six non-isomorphic subarrangements of the Catalan arrangement. They come from the subsets $\emptyset$, $\{0\}$, $\{1\}$, $\{0,1\}$, $\{-1,1\}$ and $\{-1,0,1\}$. The corresponding subarrangements are the empty arrangement, the braid arrangement, the Linial arrangement, the Shi arrangement, the interval arrangement and the Catalan arrangement, respectively. The empty arrangement is trivial, and the braid arrangement was already treated in detail in Section 2.4.1. We now have a technique that lets us talk about the remaining four arrangements under the same framework. We will do this in the remainder of this chapter.

The Linial arrangement

The Linial arrangement $\mathcal{L}_n$ consists of the hyperplanes $x_i - x_j = 1$ for $1 \leq i < j \leq n$. This arrangement was first considered by Linial and Ravid. It was later studied by Athanasiadis [5] and Postnikov and Stanley [51], who independently computed the characteristic polynomial of $\mathcal{L}_n$:

$$\chi_{\mathcal{L}_n}(q) = \frac{q}{2^n} \sum_{k=0}^{n} \binom{n}{k} (q-k)^{n-1}.$$

They also put the regions of $\mathcal{L}_n$ in bijection with several different sets of combinatorial objects. Perhaps the simplest such set is the set of *alternating trees* on $[n+1]$: the trees such that every vertex is either larger or smaller than all its neighbors.

Now we present the consequences of Proposition 2.4.10, Theorem 2.4.11 and Proposition 2.4.12 for the Linial arrangement. Recall that a poset P on $[n]$ is *naturally* *labeled* if $i < j$ in P implies $i < j$ in $\mathbb{Z}^+$.

Proposition 2.4.13 *For all* $n \geq 1$ *we have*

$$q\,\overline{\chi}_{\mathcal{L}_n}(q,t) = \sum_P q^{c(P)}(t-1)^{e(P)}$$

where the sum is over all naturally labeled, graded posets P *on* $[n]$*. Here* $c(P)$ *and* $e(P)$ *denote the number of components and edges of the Hasse diagram of* P*, respectively.*

Proof. There is an obvious bijection between Hasse diagrams of naturally labeled graded posets on $[n]$ and planted graded $\{1\}$-graphs on $[n]$. The result then follows immediately from Proposition 2.4.10. □

Theorem 2.4.14 *Let*

$$A_r(t,x) = \sum_{n \geq 0} \left(\sum_{f:[n]\to[r]} t^{id(f)} \right) \frac{x^n}{n!}.$$

where $id(f)$ *denotes the number of* inverse descents *of the word* $f(1)\ldots f(n)$*: the number of pairs* (i,j) *with* $1 \leq i < j \leq n$ *such that* $f(i) - f(j) = 1$*. Then*

$$1 + q\sum_{n\geq 1} \overline{\chi}_{\mathcal{L}_n}(q,t)\frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(t,x)}{A_{r-1}(t,x)} \right)^q.$$

Proof. This is immediate from Theorem 2.4.11. □

Recall that the *descents* of a permutation $\sigma = \sigma_1 \ldots \sigma_r \in S_r$ are the indices i such that $\sigma_i > \sigma_{i+1}$. For more information about descents, see for example [56, Section 1.3].

We call $id(f)$ the number of inverse descents, because they generalize descents in the following way. If $\pi : [r] \to [r]$ is a permutation, then $id(\pi)$ is the number of descents of the permutation π^{-1}. If, similarly, we consider the list of sets $f^{-1}(1), \ldots, f^{-1}(r)$, then $id(f)$ counts the number of occurrences of an $x \in f^{-1}(i)$ and a $y \in f^{-1}(i+1)$ such that $x > y$.

It would be very nice to compute the polynomials $A_r(t,x)$ above explicitly. We have not been able to do this. However, the special case $t=0$ is of interest, since the characteristic polynomial of $\mathcal{L}_n$ is $\chi_{\mathcal{L}_n}(q) = q\overline{\chi}_{\mathcal{L}_n}(q,0)$. In that case, we obtain the following result.

Theorem 2.4.15 *Let*

$$\frac{1+ye^{x(1+y)}}{1-y^2e^{x(1+y)}} = \sum_{r\geq 0} A_r(x)y^r. \tag{2.4.9}$$

Then we have

$$\sum_{n\geq 0} \chi_{\mathcal{L}_n}(q)\frac{x^n}{n!} = \left(\lim_{r\to\infty}\frac{A_r(x)}{A_{r-1}(x)}\right)^q.$$

In particular, if f_n is the number of alternating trees on $[n+1]$, we have

$$\sum_{n\geq 0}(-1)^n f_n\frac{x^n}{n!} = \lim_{r\to\infty}\frac{A_{r-1}(x)}{A_r(x)}.$$

Proof. In view of Theorem 2.4.14 and Proposition 2.4.12, we compute $S(0,x,y)$. From (2.4.8), the coefficient of $\frac{x^n}{n!}y^r$ in $S(0,x,y)$ is equal to the number of surjective functions $f:[n]\to[r]$ with no inverse descents. These are just the non-decreasing surjective functions $f:[n]\to[r]$. For $n\geq 1$ and $r\geq 1$ there are $\binom{n-1}{r-1}$ such functions, and for $n=r=0$ there is one such function. In the other cases there are none. Therefore

$$\begin{aligned} S(0,x,y) &= 1+\sum_{n\geq 1}\sum_{r\geq 1}\binom{n-1}{r-1}\frac{x^n}{n!}y^r \\ &= 1+\sum_{n\geq 1}\frac{x^n}{n!}y(1+y)^{n-1} \\ &= \frac{1+ye^{x(1+y)}}{1+y}. \end{aligned}$$

Proposition 2.4.12 then implies that

$$A(0,x,y) = \frac{1+ye^{x(1+y)}}{1-y^2e^{x(1+y)}},$$

in agreement with (2.4.9), and the theorem follows. □

The Shi arrangement

The Shi arrangement $\mathcal{S}_n$ consists of the hyperplanes $x_i - x_j = 0, 1$ for $1 \leq i < j \leq n$. Shi ([54, Chapter 7], [55]) first considered this arrangement, and showed that it has $(n+1)^{n-1}$ regions. Headley ([27, Chapter VI], [29]) later computed the characteristic polynomial of $\mathcal{S}_n$:

$$\chi_{\mathcal{S}_n}(q) = q(q-n)^{n-1}.$$

Stanley [57],[58] gave a nice bijection between regions of the Shi arrangement and parking functions of length n. Parking functions were first introduced by Konheim and Weiss [36]; for more information about them, see [60, Exercise 5.49].

For the Shi arrangement, we can say the following.

Theorem 2.4.16 *Let*

$$A_r(x) = \sum_{n=0}^{r} (r-n)^n \frac{x^n}{n!}.$$

Then we have

$$\sum_{n \geq 0} \chi_{\mathcal{S}_n}(q) \frac{x^n}{n!} = \left(\lim_{r \to \infty} \frac{A_r(x)}{A_{r-1}(x)} \right)^q.$$

In particular, we have

$$\sum_{n \geq 0} (-1)^n (n+1)^{n-1} \frac{x^n}{n!} = \lim_{r \to \infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

Proof. We proceed in the same way that we did in Theorem 2.4.15. In this case, we need to compute the number of surjective functions $f : [n] \to [r]$ such that $f(i) - f(j)$ is never equal to 0 or 1 for $i < j$. These are just the surjective, strictly increasing functions. There is only one of them when $n = r$, and there are none when $n \neq r$. Hence

$$S(0, x, y) = \sum_{n \geq 0} \frac{x^n}{n!} y^n = e^{xy}.$$

The rest follows easily by computing $A(0, x, y)$ and $A_r(x)$ explicitly. □

The interval arrangement

The interval arrangement $\mathcal{I}_n$ consists of the hyperplanes $x_i - x_j = -1, 1$ for $1 \leq i < j \leq n$. A *semiorder on* $[n]$ is a poset P on $[n]$ for which there exist n unit intervals $I_1, \ldots, I_n$ of $\mathbb{R}$, such that $i < j$ in P if and only if I_i is disjoint from I_j and to the left of it. It is known [53] that a poset is a semiorder if and only if it does not contain a subposet isomorphic to $\mathbf{3}+\mathbf{1}$ or $\mathbf{2}+\mathbf{2}$. We are interested in semiorders because the number of regions of $\mathcal{I}_n$ is equal to the number of semiorders on $[n]$. [51], [57]

Theorem 2.4.17 *Let*

$$\frac{1-y+ye^x}{1-y+y^2-y^2e^x} = \sum_{r\geq 0} A_r(x)y^r.$$

Then we have

$$\sum_{n\geq 0} \chi_{\mathcal{I}_n}(q)\frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(x)}{A_{r-1}(x)}\right)^q.$$

In particular, if i_n is the number of semiorders on $[n]$, we have

$$\sum_{n\geq 0} (-1)^n i_n \frac{x^n}{n!} = \lim_{r\to\infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

Proof. In this case, $S(0,x,y)$ counts surjective functions $f : [n] \to [r]$ such that $f(i) - f(j)$ is never equal to 1 for $i \neq j$. Such a function has to be constant; so it can only exist (and is unique) if $n \geq 1$ and $r = 1$ or if $n = r = 0$. Thus

$$S(0,x,y) = 1 + (e^x - 1)y$$

and the rest follows easily. □

The Catalan arrangement

The Catalan arrangement C_n consists of the hyperplanes $x_i - x_j = -1, 0, 1$ for $1 \leq i < j \leq n$. Stanley [57] observed that the number of regions of this arrangement is $n!C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. For (much) more information on the Catalan numbers, see [60, Chapter 6], especially Exercise 6.19.

Theorem 2.4.18 *Let*

$$A_r(x) = \sum_{n=0}^{\lfloor \frac{r+1}{2} \rfloor} \binom{r-n+1}{n} x^n.$$

Then we have

$$\sum_{n\geq 0} \chi_{C_n}(q) \frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(x)}{A_{r-1}(x)} \right)^q.$$

In particular,

$$\frac{\sqrt{1+4x}-1}{2x} = \sum_{n\geq 0} (-1)^n C_n x^n = \lim_{r\to\infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

Proof. There are no surjective functions $f : [n] \to [r]$ such that $f(i) - f(j)$ is never equal to $-1, 0$ or 1 for $i \neq j$, unless $n = r = 0$ or $n = r = 1$. Thus $S(x, y, 0) = 1 + xy$. The rest of the proof is easy. □

The polynomial $A_r(x)$ is a simple transformation of the *Fibonacci polynomial*. The number of words of length r, consisting of 0's and 1's, which do not contain two consecutive 1's, is equal to F_{r+2}, the $(r+2)$-th Fibonacci number. It is easy to see that the polynomial $A_r(x)$ counts those words according to the number of 1's they contain.

Chapter 3

Semimatroids and their Tutte polynomials

3.1 Introduction

In this chapter, we define a class of objects called *semimatroids*. A semimatroid can be thought of as a matroid-theoretic abstraction of the dependence properties of an affine hyperplane arrangement. Many properties of hyperplane arrangements are not really facts about the arrangements themselves, but about their underlying matroidal structure. Therefore, the study of such properties can be carried out much more naturally and elegantly in the setting of semimatroids.

In Section 3.2 we define semimatroids, and show how we can think of a hyperplane arrangement as a semimatroid. The following sections provide different ways of thinking about semimatroids. Section 3.3 shows how a semimatroid "extends" to a matroid and determines a modular ideal inside it. The semimatroid can be recovered from the matroid and its modular ideal. Section 3.4 describes the close relationship between semimatroids and strong maps. Semimatroids are described in terms of elementary preimages and single-element coextensions. Section 3.5 gives a bijection between semimatroids and pointed matroids. Section 3.6 gives a new characterization of geometric semilattices as posets of flats of semimatroids. The final sections are geared towards the study of the Tutte polynomial. Section 3.7 defines the concept of duality, deletion and contraction for semilattices. Section 3.8 defines the Tutte polynomial of a semimatroid, and shows that it is the unique Tutte-Grothendieck invariant for the class of semimatroids. Finally, Section 3.9 gives a combinatorial interpretation for the non-negative coefficients of the Tutte polynomial of a semimatroid.

Throughout Chapter 3, we will assume some familiarity with the basic concepts of matroid theory. For instance, Chapter 1 of [47] should be enough to understand most of this chapter. Some familiarity with strong maps and geometric semilattices would also be useful, but not necessary.

3.2 Semimatroids

Definition 3.2.1 *A* semimatroid *is a triple* $(S, \mathcal{C}, r_{\mathcal{C}})$ *consisting of a finite set* S*, a non-empty simplicial complex* $\mathcal{C}$ *on* S*, and a function* $r_{\mathcal{C}} : \mathcal{C} \to \mathbb{N}$*, satisfying the following five conditions.*

(R1) *If* $X \in \mathcal{C}$*, then* $0 \leq r_{\mathcal{C}}(X) \leq |X|$*.*

(R2) *If* $X, Y \in \mathcal{C}$ *and* $X \subseteq Y$*, then* $r_{\mathcal{C}}(X) \leq r_{\mathcal{C}}(Y)$*.*

(R3) *If* $X, Y \in \mathcal{C}$ *and* $X \cup Y \in \mathcal{C}$*, then* $r_{\mathcal{C}}(X) + r_{\mathcal{C}}(Y) \geq r_{\mathcal{C}}(X \cup Y) + r_{\mathcal{C}}(X \cap Y)$*.*

(CR1) *If* $X, Y \in \mathcal{C}$ *and* $r_{\mathcal{C}}(X) = r_{\mathcal{C}}(X \cap Y)$*, then* $X \cup Y \in \mathcal{C}$*.*

(CR2) *If* $X, Y \in \mathcal{C}$ *and* $r_{\mathcal{C}}(X) < r_{\mathcal{C}}(Y)$*, then* $X \cup y \in \mathcal{C}$ *for some* $y \in Y - X$*.*

We call S, $\mathcal{C}$ and $r_{\mathcal{C}}$ the *ground set*, *collection of central sets* and *rank function* of the semimatroid $(S, \mathcal{C}, r_{\mathcal{C}})$, respectively. Sometimes we will slightly abuse notation and denote the semimatroid $\mathcal{C}$, when its ground set and rank function are clear. We will denote subsets of S by upper case letters, and elements of S by lower case letters.

We will need the fact that semimatroids satisfy a "local" version of (R1) and (R2) and a stronger version of (CR1) and (CR2), as follows.

(R2') If $X \cup x \in \mathcal{C}$ then $r_{\mathcal{C}}(X \cup x) - r_{\mathcal{C}}(X) = 0$ or 1.

(CR1') If $X, Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X) = r_{\mathcal{C}}(X \cap Y)$, then $X \cup Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup Y) = r_{\mathcal{C}}(Y)$.

(CR2') If $X, Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X) < r_{\mathcal{C}}(Y)$, then $X \cup y \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup y) = r_{\mathcal{C}}(X) + 1$ for some $y \in Y - X$.

Proof of (R2'). From (R2) we know that $r_{\mathcal{C}}(X \cup x) \geq r_{\mathcal{C}}(X)$. From (R3) we know that $r_{\mathcal{C}}(X \cup x) - r_{\mathcal{C}}(X) \leq r_{\mathcal{C}}(x) - r_{\mathcal{C}}(\emptyset)$, and this is 0 or 1 by (R1). □

Proof of (CR1'). The hypotheses imply that $X \cup Y \in \mathcal{C}$. Then (R2) says that $r_{\mathcal{C}}(Y) \leq r_{\mathcal{C}}(X \cup Y)$, while (R3) says that $r_{\mathcal{C}}(Y) \geq r_{\mathcal{C}}(X \cup Y)$. □

Proof of (CR2'). By applying (CR2) repeatedly, we see that we can keep on adding elements $y_1, \ldots, y_k$ of Y to the set X, until we reach a set $X \cup y_1 \cup \cdots \cup y_k \in \mathcal{C}$ such that $r_{\mathcal{C}}(X \cup y_1 \cup \cdots \cup y_k) = r_{\mathcal{C}}(Y)$. Now we claim that $r_{\mathcal{C}}(X \cup y_i) = r_{\mathcal{C}}(X) + 1$ for some i.

If that was not the case then, since $r_{\mathcal{C}}(X \cup y_1) = r_{\mathcal{C}}(X)$, (CR1') applies to $X \cup y_1$ and $X \cup y_2$. Therefore $X \cup y_1 \cup y_2 \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup y_1 \cup y_2) = r_{\mathcal{C}}(X \cup y_2) = r_{\mathcal{C}}(X)$. Then (CR1') applies to $X \cup y_1 \cup y_2$ and $X \cup y_3$, so $X \cup y_1 \cup y_2 \cup y_3 \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup y_1 \cup y_2 \cup y_3) = r_{\mathcal{C}}(X)$. Continuing in this way, we conclude that $X \cup y_1 \cup \cdots \cup y_k \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup y_1 \cup \cdots \cup y_k) = r_{\mathcal{C}}(X)$, a contradiction. □

The following proposition shows the connection between semimatroids and hyperplane arrangements, and explains the name we have given to $\mathcal{C}$.

Proposition 3.2.2 *Let $\mathcal{A}$ be an affine hyperplane arrangement in a vector space V. Let $\mathcal{C}_{\mathcal{A}}$ be the collection of central subarrangements of $\mathcal{A}$, and let $r_{\mathcal{A}}$ be the rank function of $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{C}_{\mathcal{A}}, r_{\mathcal{A}})$ is a semimatroid.*

Proof. Say the vector space V is n-dimensional. To each hyperplane $H_i \in \mathcal{A}$ we can associate a linear functional $L_i \in V^*$ and a constant c_i, so that H_i is the set of points $x \in V$ such that $L_i(x) = c_i$. Then, as remarked in Section 2.2, the rank of a central subset $\{H_{i_1}, \ldots, H_{i_k}\} \in \mathcal{C}_{\mathcal{A}}$ is equal to the rank of the set $\{L_{i_1}, \ldots, L_{i_k}\}$ in V^*.

From this point of view, axioms (R1), (R2), (R3) are standard facts of linear algebra applied to the vector space V^*. We now check axioms (CR1) and (CR2).

To check axiom (CR1), assume that $X, Y \in \mathcal{C}$ and $r_{\mathcal{A}}(X) = r_{\mathcal{A}}(X \cap Y)$. Let $A = \cap X$ and $B = \cap Y$. Since $X \cap Y \subseteq X$ and $r_{\mathcal{A}}(X \cap Y) = r_{\mathcal{A}}(X)$, we must have $\cap(X \cap Y) = \cap X = A$. Also, $X \cap Y \subseteq Y$ implies $\cap(X \cap Y) \supseteq \cap Y = B$. Therefore $A \supseteq B$, and every hyperplane in $X \cup Y$ contains B. It follows that $X \cup Y \in \mathcal{C}$.

To check axiom (CR2), assume that $X, Y \in \mathcal{C}$ and $r_{\mathcal{A}}(X) < r_{\mathcal{A}}(Y)$. Let $L_X = \{L_i \mid H_i \in X\}$ and define similarly L_Y. Since $\text{rank}(L_Y) > \text{rank}(L_X)$, there exists a vector $L \in L_Y$, corresponding to a hyperplane $y \in Y$, which is not in the span of L_X. Thus y has a non-empty intersection with $\cap X$. $\square$

Semimatroids, like matroids, have several equivalent definitions. In their context, it is possible to talk about flats, independent sets, spanning sets, bases, circuits, and most other basic matroid concepts. We will use the rank function approach of Definition 3.2.1 throughout most of our treatment of semimatroids. We will also need some facts about the closure approach, which we now present.

Definition 3.2.3 *For a semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ and a set $X \in \mathcal{C}$, the* closure of X in $\mathcal{C}$ *is* $\text{cl}_{\mathcal{C}}(X) = \{x \in S \mid X \cup x \in \mathcal{C}, r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X)\}$.

We will sometimes drop the subscript and write $\text{cl}(X)$ instead of $\text{cl}_{\mathcal{C}}(X)$ when it causes no confusion.

Proposition 3.2.4 *The closure operator of a semimatroid satisfies the following properties, for all $X, Y \in \mathcal{C}$ and $x, y \in S$.*

(CLR1) $\text{cl}(X) \in \mathcal{C}$ *and* $r_{\mathcal{C}}(\text{cl}(X)) = r_{\mathcal{C}}(X)$.

(CL1) $X \subseteq \text{cl}(X)$.

(CL2) *If* $X \subseteq Y$ *then* $\text{cl}(X) \subseteq \text{cl}(Y)$.

(CL3) $\text{cl}(\text{cl}(X)) = \text{cl}(X)$.

(CL4) *If* $X \cup x \in \mathcal{C}$ *and* $y \in \text{cl}(X \cup x) - \text{cl}(X)$, *then* $X \cup y \in \mathcal{C}$ *and* $x \in \text{cl}(X \cup y)$.

Proof. To check (CLR1), let $\text{cl}(X) = \{x_1, \ldots, x_k\}$. We repeat the argument of the proof of (CL2'). Since $r_{\mathcal{C}}(X \cup x_1) = r_{\mathcal{C}}(X)$, (CR1') applies to $X \cup x_1$ and $X \cup x_2$, so $X \cup x_1 \cup x_2 \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x_1 \cup x_2) = r_{\mathcal{C}}(X)$. (CR1') then applies to $X \cup x_1 \cup x_2$ and $X \cup x_3$, so $X \cup x_1 \cup x_2 \cup x_3 \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x_1 \cup x_2 \cup x_3) = r_{\mathcal{C}}(X)$. Continuing in this way, we conclude that $X \cup x_1 \cup \cdots \cup x_k \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x_1 \cup \cdots \cup x_k) = r_{\mathcal{C}}(X)$.

(CL1) is trivial.

To check (CL2), let $x \in \text{cl}(X)$. Then $X \cup x \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X)$. Applying (CR1') to $X \cup x$ and Y, we conclude that $Y \cup x \in \mathcal{C}$ and $r_{\mathcal{C}}(Y \cup x) = r_{\mathcal{C}}(Y)$. Therefore $x \in \text{cl}(Y)$.

We know that $\mathrm{cl}(X) \subseteq \mathrm{cl}(\mathrm{cl}(X))$; so to prove (CL3) it suffices to check the reverse inclusion. Let $x \in \mathrm{cl}(\mathrm{cl}(X))$. Then $\mathrm{cl}(X) \cup x \in \mathcal{C}$ and $r_{\mathcal{C}}(\mathrm{cl}(X) \cup x) = r_{\mathcal{C}}(\mathrm{cl}(X)) = r_{\mathcal{C}}(X)$. Therefore, since $\mathrm{cl}(X) \cup x \supseteq X \cup x \supseteq X$, we have $X \cup x \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X)$ also; *i.e.*, $x \in \mathrm{cl}(X)$.

Finally, we check (CL4). The assumption that $y \in \mathrm{cl}(X \cup x)$ implies that $X \cup x \cup y \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup x \cup y) = r_{\mathcal{C}}(X \cup x) \leq r_{\mathcal{C}}(X) + 1$. Since $X \cup y \in \mathcal{C}$, the assumption that $y \notin \mathrm{cl}(X)$ implies that $r_{\mathcal{C}}(X) + 1 = r_{\mathcal{C}}(X \cup y)$. These two results together give $r_{\mathcal{C}}(X \cup x \cup y) = r_{\mathcal{C}}(X \cup y)$; *i.e.*, $x \in \mathrm{cl}(X \cup y)$. □

We will later need the following definitions.

Definition 3.2.5 *A* flat *of a semimatroid* $\mathcal{C}$ *is a set* $A \in \mathcal{C}$ *such that* $\mathrm{cl}(A) = A$. *The* poset of flats $K(\mathcal{C})$ *of a semimatroid* $\mathcal{C}$ *is the set of flats of* $\mathcal{C}$*, ordered by containment.*

3.3 Modular ideals

In the following sections, we will present bijections between the class of semimatroids and other important classes of objects. Figure 3.4 at the end of Section 3.4 should be useful in understanding these bijections, and is worth keeping in mind especially in Sections 3.3, 3.4 and 3.5.

In this section, we show that a semimatroid is essentially equivalent to a pair $(M, \mathcal{I})$ of a matroid M and one of its modular ideals $\mathcal{I}$.

We start by showing how we can naturally construct a matroid $M_{\mathcal{C}}$ from a given semimatroid $(S, \mathcal{C}, r_{\mathcal{C}})$, by extending the rank function $r_{\mathcal{C}}$ from $\mathcal{C}$ to 2^S.

Proposition 3.3.1 *Let* $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ *be a semimatroid. For each subset* $X \subseteq S$*, let* $r(X) = \max\{r_{\mathcal{C}}(Y) \mid Y \subseteq X\, ,\, Y \in \mathcal{C}\}$. *Then* r *is the rank function of a matroid* $M_{\mathcal{C}} = (S, r)$.

Proof. It is clear, but worth remarking explicitly, that $r(X) = r_{\mathcal{C}}(X)$ if $X \in \mathcal{C}$. It will be most convenient to check the three "local" axioms (R1')-(R3') for the rank function of a matroid.[16] Let $X \subseteq S$ and $a, b \in S$ be arbitrary.

(R1') $r(\emptyset) = 0$.

This is trivial.

(R2') $r(X \cup a) - r(X) = 0$ or 1.

This is easy. It is immediate from the definition that $r(X \cup a) \geq r(X)$. Now let $r(X \cup a) = r_{\mathcal{C}}(Y)$ for $Y \subseteq X \cup a$, $Y \in \mathcal{C}$. Then $Y - a \subseteq X$ is also in $\mathcal{C}$, so $r(X) \geq r_{\mathcal{C}}(Y - a) \geq r_{\mathcal{C}}(Y) - 1 = r(X \cup a) - 1$.

(R3') If $r(X \cup a) = r(X \cup b) = r(X)$, then $r(X \cup a \cup b) = r(X)$.

This takes more work. Assume that $r(X \cup a) = r(X \cup b) = r(X) = s$ but $r(X \cup a \cup b) = s + 1$. Let $W \subseteq X \cup a \cup b$, $W \in \mathcal{C}$ be such that $r_{\mathcal{C}}(W) = s + 1$. Notice that W must contain a; otherwise we would have $W \subseteq X \cup b$ and $r_{\mathcal{C}}(W) > r(X \cup b)$. Similarly, W contains b. So let $W = Z \cup a \cup b$; clearly $Z \subseteq X$.

We have $s + 1 = r_{\mathcal{C}}(Z \cup a \cup b) \leq r_{\mathcal{C}}(Z \cup a) + 1 \leq r(X \cup a) + 1 = s + 1$. Therefore $r_{\mathcal{C}}(Z \cup a) = s$. Similarly, $r_{\mathcal{C}}(Z \cup b) = s$. Then, by the submodularity of $r_{\mathcal{C}}$, $r_{\mathcal{C}}(Z) = s - 1$.

Now, since $r(X) = s$, we can find $V \subseteq X$, $V \in \mathcal{C}$ such that $r_{\mathcal{C}}(V) = s$. So we have $V, Z \in \mathcal{C}$ with $s = r_{\mathcal{C}}(V) > r_{\mathcal{C}}(Z) = s - 1$. By (CR2'), we can add an element of V to Z and obtain a set $Y \in \mathcal{C}$ with $X \supseteq Y \supseteq Z$ such that $r_{\mathcal{C}}(Y) = s$. Notice that $Z \cup a \subseteq Y \cup a \subseteq X \cup a$ and $r(Z \cup a) = r(X \cup a) = s$. Thus $r(Y \cup a) = s$. Similarly, $r(Y \cup b) = s$ and $r(Y \cup a \cup b) = s + 1$.

Now we have $Y, Z \cup a \cup b \in \mathcal{C}$ with $s + 1 = r_{\mathcal{C}}(Z \cup a \cup b) > r_{\mathcal{C}}(Y) = s$. Once again, (CR2') guarantees that we can add an element of $Z \cup a \cup b$ to Y to obtain an element of rank $s + 1$ in $\mathcal{C}$. But $Z \subseteq Y$, so the only elements of $Z \cup a \cup b$ which may not be in Y are a and b. Also, we saw that $r(Y \cup a) = r(Y \cup b) = s$. This is a contradiction. □

The following definitions will be important to us.

Definition 3.3.2 *A pair* $\{X, Y\}$ *of subsets of* S *is a* modular pair *of the matroid* $M = (S, r)$ *if* $r(X) + r(Y) = r(X \cup Y) + r(X \cap Y)$.

Definition 3.3.3 *[22] A* modular ideal $\mathcal{I}$ *of a matroid* $M = (S, r)$ *is a non-empty collection of subsets of* S *satisfying the following three conditions.*

(MI1) *$\mathcal{I}$ is a simplicial complex.*

(MI2) *$\{a\} \in \mathcal{I}$ for every non-loop a of M.*

(MI3) *If $\{X, Y\}$ is a modular pair in M and $X, Y \in \mathcal{I}$, then $X \cup Y \in \mathcal{I}$.*

Proposition 3.3.4 *For any semimatroid $(S, \mathcal{C}, r_{\mathcal{C}})$, the collection $\mathcal{C}$ is a modular ideal of $M_{\mathcal{C}}$.*

Proof. We denote the rank function of $M_{\mathcal{C}}$ by r and the rank function of $\mathcal{C}$ by $r_{\mathcal{C}}$. Of course, $r_{\mathcal{C}}$ is just the restriction of r to $\mathcal{C}$.

Axioms (MI1) and (MI2) of a modular ideal are satisfied trivially. We reformulate (MI3) as follows:

(MI3) If $A, B, C \subseteq S$ are pairwise disjoint, $A \cup B, A \cup C \in \mathcal{C}$ and $r(A \cup B \cup C) - r(A \cup B) = r(A \cup C) - r(A)$, then $A \cup B \cup C \in \mathcal{C}$.

We can assume that B and C are non-empty; if one of them is empty, the claim is trivial. We prove (MI3) by induction on $|B| + |C|$.

The first case is $|B| + |C| = 2$; let $B = \{b\}$ and $C = \{c\}$. First assume that $r(A \cup b)$ and $r(A \cup c)$ are different; say, $r_{\mathcal{C}}(A \cup b) < r_{\mathcal{C}}(A \cup c)$. By (CR2), we can add some element of $A \cup c$ to $A \cup b$ and obtain a set in $\mathcal{C}$. This element can only be c, so $A \cup b \cup c \in \mathcal{C}$.

Assume then that $r_{\mathcal{C}}(A \cup b) = r_{\mathcal{C}}(A \cup c) = s$. If $r_{\mathcal{C}}(A) = s$, (CR1) implies that $A \cup b \cup c \in \mathcal{C}$. Assume then that $r_{\mathcal{C}}(A) = s - 1$, and therefore $r(A \cup b \cup c) = s + 1$ by hypothesis. There is a subset of $A \cup b \cup c$ in $\mathcal{C}$ of rank $s + 1$; since it cannot be contained in $A \cup b$ or $A \cup c$, it must be of the form $B \cup b \cup c$ for some $B \subseteq A$. But then we have $r_{\mathcal{C}}(A \cup b) < r_{\mathcal{C}}(B \cup b \cup c)$. By (CR2), we can add some element of $B \cup b \cup c$ to $A \cup b$ and obtain a set in $\mathcal{C}$. This element can only be c, so $A \cup b \cup c \in \mathcal{C}$.

Having established the base case of the induction, we proceed with the inductive step. Assume that $|B| + |C| \geq 3$ and, without loss of generality, $|B| \geq 2$. Let $b \in B$. Applying the submodularity of r twice, we get that $d = r(A \cup B \cup C) - r(A \cup B) \geq r(A \cup b \cup C) - r(A \cup b) \geq r(A \cup C) - r(A) = d$. It follows that $r(A \cup b \cup C) - r(A \cup b) = d$ also.

We can apply the induction hypothesis to the sets $A, \{b\}, C$, since $A \cup b, A \cup C \in \mathcal{C}$ and $|\{b\}| + |C| < |B| + |C|$. We conclude that $A \cup b \cup C \in \mathcal{C}$. We can then apply the induction hypothesis to the sets $A \cup b, B - b, C$, since $A \cup B, A \cup b \cup C \in \mathcal{C}$ and $|B - b| + |C| < |B| + |C|$. We conclude that $A \cup B \cup C \in \mathcal{C}$, as desired. □

Propositions 3.3.1 and 3.3.4 show us how to obtain a pair $(M, \mathcal{I})$ of a matroid M and one of its modular ideals $\mathcal{I}$, given a semimatroid $\mathcal{C}$. Now we show that it is possible to recover $\mathcal{C}$ from the pair $(M, \mathcal{I})$.

Proposition 3.3.5 *Let $M = (S, r)$ be a matroid, and let $\mathcal{I}$ be a modular ideal of M. Let $r_{\mathcal{I}}$ be the restriction of the rank function of M to $\mathcal{I}$. Then $(S, \mathcal{I}, r_{\mathcal{I}})$ is a semimatroid.*

Proof. The rank function $r_{\mathcal{I}}$ inherits axioms (R1)-(R3) from r_M. (CR1) is easy. If $X, Y \in \mathcal{I}$ and $r_{\mathcal{I}}(X) = r_{\mathcal{I}}(X \cap Y)$, then $r(Y) = r(X \cup Y)$ by submodularity. Thus $\{X, Y\}$ is a modular pair in M, and $X \cup Y \in \mathcal{I}$.

Now we check (CR2). We start by showing that $\mathcal{I}$ must contain every independent set of M. In fact, assume that I is a minimal independent set which is not in $\mathcal{I}$. Since $\mathcal{I}$ contains all non-loop elements, I has at least two elements a and b. Then $\mathcal{I}$ contains the modular pair $\{I - a, I - b\}$, so it contains their union I, a contradiction.

Now take $X, Y \in \mathcal{I}$ with $|X| < |Y|$, and pick $y \in Y$ such that $r(X \cup y) = r(X) + 1$. Let X' be an independent subset of X of rank $r(X)$; then $X' \cup y$ is an independent set of rank $r(X) + 1$. Therefore $\mathcal{I}$ contains the modular pair $\{X' \cup y, X\}$, so it contains their union $X \cup y$. □

Theorem 3.3.6 *Let S be a finite set. Let* $\text{Semimat}(S)$ *be the set of semimatroids on S. Let* $\text{MatId}(S)$ *be the set of pairs $(M, \mathcal{I})$ of a matroid M on S and a modular ideal $\mathcal{I}$ of M.*

1. *The assignment $(S, \mathcal{C}, r_{\mathcal{C}}) \mapsto (M_{\mathcal{C}}, \mathcal{C})$ is a map* $\text{Semimat}(S) \to \text{MatId}(S)$.

2. *The assignment $(M, \mathcal{I}) \mapsto (S, \mathcal{I}, r_{\mathcal{I}})$ is a map* $\text{MatId}(S) \to \text{Semimat}(S)$.

3. *The two maps above are inverses, and give a one-to-one correspondence between* $\mathrm{Semimat}(S)$ *and* $\mathrm{MatId}(S)$.

Proof. The first and second parts are restatements of Propositions 3.3.1 and 3.3.4 and Proposition 3.3.5, respectively.

Denote the maps $\mathrm{Semimat}(S) \to \mathrm{MatId}(S)$ and $\mathrm{MatId}(S) \to \mathrm{Semimat}(S)$ above by f and g respectively. It is immediate that $g \circ f$ is the identity map in $\mathrm{Semimat}(S)$. To check that $f \circ g$ is the identity map in $\mathrm{MatId}(S)$, we need to show the following. Given a matroid $M = (S, r)$ and a modular ideal $\mathcal{I}$ of M, $r(X) = \max\{r(Y) \mid Y \subseteq X\,,\, Y \in \mathcal{I}\}$ for all $X \subseteq S$. But this is easy: it is clear that $r(X) \geq \max\{r(Y) \mid Y \subseteq X\,,\, Y \in \mathcal{I}\}$. Equality is attained because X has an independent subset X' of rank $r(X)$; since X' is independent, it is in $\mathcal{I}$. □

Before we continue our analysis, we state explicitly a simple property of semimatroids and modular ideals which is implicit in the proofs above.

In a semimatroid $(S, \mathcal{C}, r_{\mathcal{C}})$, all the maximal sets in $\mathcal{C}$ have the same rank, which we denote $r_{\mathcal{C}}$. In a modular ideal $\mathcal{I}$ of a matroid $M = (S, r)$, all the maximal sets have maximum rank $r = r(S)$.

3.4 Elementary preimages and single-element coextensions

Now we show that a semimatroid is also equivalent to a pair (M, M') of a matroid M and one of its rank-increasing single-element coextensions M'. To do it, we outline the correspondence between the modular ideals, the elementary preimages and the rank-increasing single-element coextensions of a matroid.

This correspondence is just the dual of the well understood correspondence between the modular filters, the elementary quotients, and the rank-preserving single-element extensions of a matroid [19], [22], [39]. Therefore we omit all the proofs of these results, and refer the reader to the relevant literature.

Definition 3.4.1 *A* quotient map $N \to M$ *is a pair of matroids* M, N *on the same ground set such that every flat of* M *is a flat of* N*.*

There are several other equivalent definitions of quotient maps, including the following.

Proposition 3.4.2 *[39, Proposition 8.1.6] Let* M *and* N *be two matroids on the set* S*. The following are equivalent:*
(i) $N \to M$ *is a quotient map.*
(ii) For any $A \subseteq S$*,* $\mathrm{cl}_N(A) \subseteq \mathrm{cl}_M(A)$*.*
(iii) For any $A \subseteq B \subseteq S$*,* $r_N(B) - r_N(A) \geq r_M(B) - r_M(A)$*.*

Definition 3.4.3 *An* elementary quotient map *is a quotient map* $N \to M$ *such that* $r(N) - r(M) = 0$ *or* 1*.*

We will focus our attention on elementary quotient maps. Their importance is the following. Perhaps the most useful notion of a morphism in the category of matroids is that of a *strong map*. Every strong map between matroids can be regarded essentially as a quotient map, followed by an embedding of a submatroid into a matroid. Also, every quotient map can be factored into a sequence of elementary quotient maps. Therefore, elementary quotient maps are essentially the building blocks of strong maps. For more information on this topic, we refer the reader to [39].

Definition 3.4.4 *An* elementary preimage *of a matroid* M *is a matroid* N *on the same ground set such that* $N \to M$ *is an elementary quotient map.*

The following proposition explains the relevance of elementary preimages and quotient maps in our investigation.

Theorem 3.4.5 *[22, Proposition 6.5] Let* $M = (S, r_M)$ *be a matroid. Let* $\mathrm{Ideal}(M)$ *be the set of modular ideals of* M *and let* $\mathrm{Preim}(M)$ *be the set of elementary preimages of* M*.*

1. *Given* $\mathcal{I} \in \mathrm{Ideal}(M)$*, define the rank function* $r_N : 2^S \to \mathbb{N}$ *by:*

$$r_N(A) = \begin{cases} r_M(A) & \text{if } A \in \mathcal{I} \\ r_M(A) + 1 & \text{if } A \notin \mathcal{I} \end{cases}$$

   Then $N = (S, r_N)$ *is a matroid, and* $N \in \mathrm{Preim}(M)$.

2. *Given* $N \in \mathrm{Preim}(M)$*, let* $\mathcal{I} = \{A \in S : r_M(A) = r_N(A)\}$. *Then* $\mathcal{I} \in \mathrm{Ideal}(M)$.

3. *The two maps* $\mathrm{Ideal}(M) \to \mathrm{Preim}(M)$ *and* $\mathrm{Preim}(M) \to \mathrm{Ideal}(M)$ *defined above are inverses. They establish a one-to-one correspondence between* $\mathrm{Ideal}(M)$ *and* $\mathrm{Preim}(M)$.

Corollary 3.4.6 *Given a finite set* S*, let* $\mathrm{MatPreim}(S)$ *be the set of pairs* (M, N) *of a matroid* M *on* S *and one of its elementary preimages* N*. Then there are one-to-one correspondences between* $\mathrm{Semimat}(S)$*,* $\mathrm{MatId}(S)$ *and* $\mathrm{MatPreim}(S)$.

Proof. Combine Theorems 3.3.6 and 3.4.5. □.

Definition 3.4.7 *Let* M *be a matroid on the ground set* S *and let* p *be an element not in* S*. A* single-element coextension *of* M *by* p *is a matroid* $\widetilde{N}$ *on the set* $S \cup p$ *such that* $M = \widetilde{N}/p$. $\widetilde{N}$ *is* rank-increasing *if* $r(\widetilde{N}) > r(M)$.

It is worth remarking that most single-element coextensions of M by p are rank-increasing. The only one which is not rank-increasing is the matroid $\widetilde{N}$ on $S \cup p$ such that $r_{\widetilde{N}}(A \cup p) = r_{\widetilde{N}}(A) = r_M(A)$ for all $A \subseteq S$; *i.e.*, the one where p is a loop.

Theorem 3.4.8 *[39, dual of Theorem 8.3.2] Let* M *be a matroid and* p *be an element not in its ground set. Let* $\mathrm{Coext}(M, p)$ *be the set of rank-increasing single-element coextensions of* M *by* p.

1. *Given* $N \in \mathrm{Preim}(M)$*, define* $r_{\widetilde{N}} : 2^{S \cup p} \to \mathbb{N}$ *by*

$$\begin{aligned} r_{\widetilde{N}}(A) &= r_N(A) \\ r_{\widetilde{N}}(A \cup p) &= r_M(A) + 1 \end{aligned}$$

for $A \subseteq S$*. Then* $\widetilde{N} = (S \cup p, r_{\widetilde{N}})$ *is a matroid, and* $\widetilde{N} \in \mathrm{Coext}(M,p)$*.*

2. *If* $\widetilde{N} \in \mathrm{Coext}(M,p)$*, then the matroid* $N = \widetilde{N} - p$ *is in* $\mathrm{Preim}(M)$*.*

3. *The two maps* $\mathrm{Preim}(M) \to \mathrm{Coext}(M,p)$ *and* $\mathrm{Coext}(M,p) \to \mathrm{Preim}(M)$ *defined above are inverses. They establish a one-to-one correspondence between* $\mathrm{Preim}(M)$ *and* $\mathrm{Coext}(M,p)$*.*

Corollary 3.4.9 *Given a finite set* S *and an element* $p \notin S$*, let* $\mathrm{MatCoext}(S,p)$ *be the set of pairs* $(M, \widetilde{N})$*, where* M *is a matroid on* S *and* $\widetilde{N}$ *is one of its rank-increasing single-element coextensions by* p*. Then there are one-to-one correspondences between* $\mathrm{Semimat}(S)$*,* $\mathrm{MatId}(S)$*,* $\mathrm{MatPreim}(S)$ *and* $\mathrm{MatCoext}(S,p)$*.*

Proof. Combine Theorems 3.3.6, 3.4.5 and 3.4.8. □

We briefly mention that given a matroid M, there are other objects in correspondence with the modular ideals of M. Two such examples are the *modular cocuts* of M and the *colinear subclasses* of M. They are the duals of modular cuts and linear subclasses, respectively.

A modular cocut $\mathcal{U}$ of M is a collection of circuit unions of M satisfying two conditions. First, if $U_1 \subseteq U_2$ are circuit unions and $U_2 \in \mathcal{U}$, then $U_1 \in \mathcal{U}$. Second, if $U_1, U_2 \in \mathcal{U}$ and $\{U_1, U_2\}$ is a modular pair in M, then $U_1 \cup U_2 \in \mathcal{U}$.

A colinear subclass $\mathcal{C}$ of M is a set of circuits of M such that if $C_1, C_2 \in \mathcal{C}$ and $r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$, and $C_3 \subseteq C_1 \cup C_2$ is a circuit, then $C_3 \in \mathcal{C}$.

The details and proofs of the (dual) correspondences appear in [47, Theorem 7.2.2] and [19], respectively.

We end this section by reviewing the correspondences and objects of Sections 3.3 and 3.4 with an example. Let $\mathcal{C} = (S, \mathcal{C}, r)$ be the semimatroid consisting of the set $S = [3]$, the collection of central sets $\mathcal{C} = 2^{[3]} - \{12, 123\}$, and the rank function $r(X) = |X|$ for $X \in \mathcal{C}$. It is easy to check that this is, indeed, a semimatroid. The first diagram of Figure 3-1 depicts the poset on $\mathcal{C}$ ordered by inclusion; below and to the right of each node we have written the set in $\mathcal{C}$ corresponding to it, and above and to its left we have written its rank.

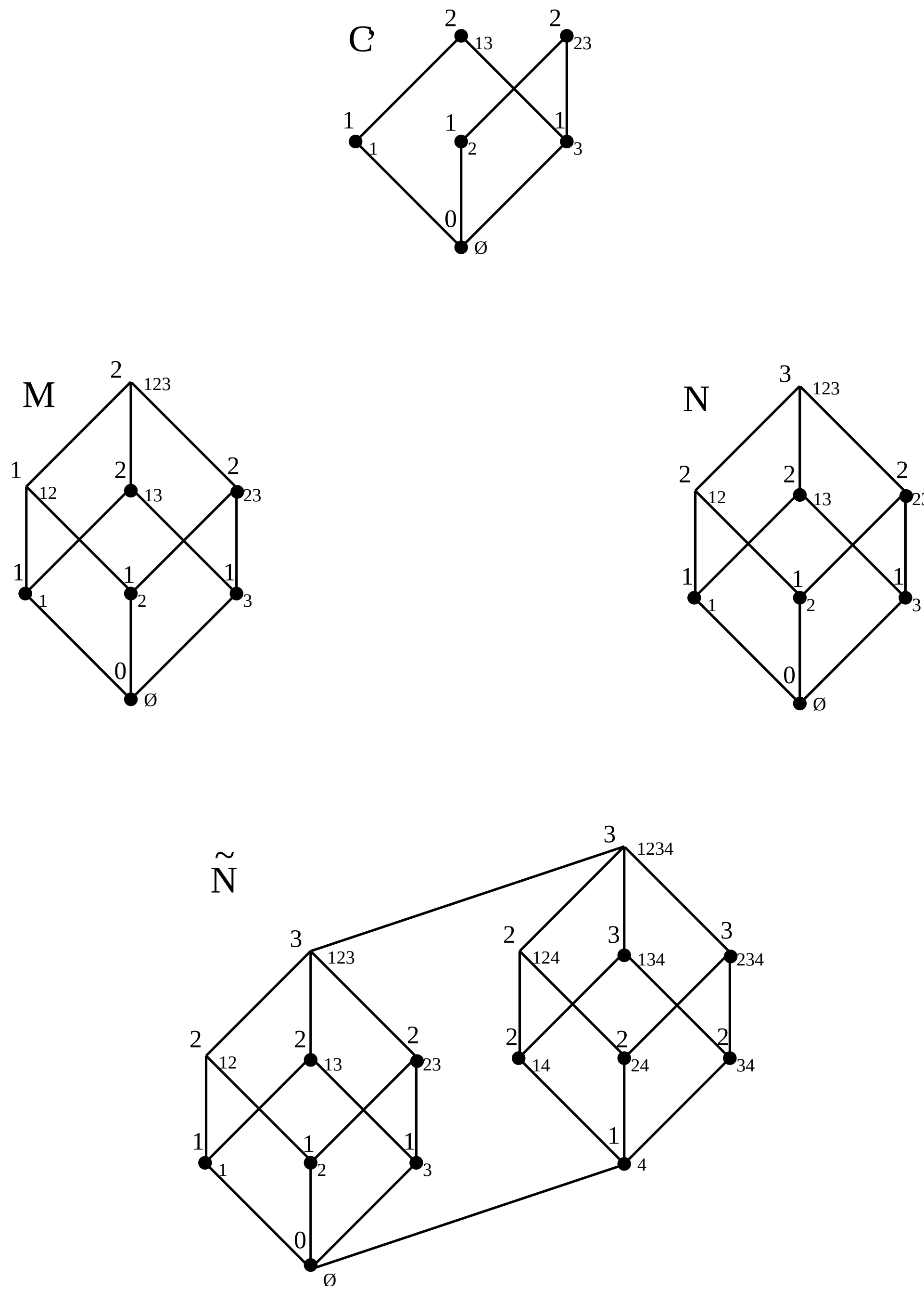

Figure 3-1: The semimatroid $\mathcal{C}$ and its corresponding matroids.

To the semimatroid $\mathcal{C}$, we have assigned a pair $(M, \mathcal{I}) \in \text{MatId}(S)$, a pair $(M, N) \in \text{MatPreim}(S)$ and a pair $(M, \widetilde{N}) \in \text{MatCoext}(S)$. To obtain the matroid M, we add the subsets of $[3]$ not in $\mathcal{C}$ to the diagram above, to get the Boolean algebra $2^{[3]}$. We have placed big nodes on the sets of this poset which are in the diagram of $\mathcal{C}$, and small nodes on the new sets. To obtain the rank function of M, we copy the rank function of $\mathcal{C}$ on the big nodes. On each small node, we put the largest number that we can find on a big node below it. The big nodes form the modular ideal $\mathcal{I}$ of M.

To obtain the matroid N, we simply leave the rank function of M fixed on the big nodes, and increase it by 1 on the little nodes.

Finally, to obtain the matroid $\widetilde{N}$, we glue two Boolean algebras $2^{[3]}$, to obtain a Boolean algebra $2^{[4]}$ on 4 elements. (We have omitted most of the "diagonal" edges of this poset for clarity.) On the lower copy of the Boolean algebra, we put the rank function of N. On the upper copy, we put the rank function of M, increased by 1.

3.5 Pointed matroids

We now establish a correspondence between semimatroids and pointed matroids.

Definition 3.5.1 *[14] A* pointed matroid *is a pair (M, p) of a matroid M and a distinguished element p of its ground set.*

Pointed matroids are a useful combinatorial tool for the study of affine hyperplane arrangements. The connection is the following. Consider an affine arrangement $\mathcal{A} = \{H_1, \ldots, H_k\}$ in $\mathbb{R}^n$, where H_i is defined by the equation $L_i(x) = c_i$.

Definition 3.5.2 *The* cone over $\mathcal{A}$ *is the arrangement $c\mathcal{A} = \{H'_1, \ldots, H'_k, H\}$ in $\mathbb{R}^{n+1}$, where H'_i is defined by the equation $L_i(x) = c_i x_{n+1}$ for $1 \leq i \leq k$, and H is the* additional hyperplane $x_{n+1} = 0$.

Being a central arrangement, $c\mathcal{A}$ has a matroid $M_{c\mathcal{A}}$ on the ground set $c\mathcal{A}$ associated to it. To the arrangement $\mathcal{A}$, we associate the pointed matroid $(M_{c\mathcal{A}}, H)$.

Theorem 3.5.3 *Let S be a set and let $p \notin S$. Let* $\mathrm{Pointedmat}(S,p)$ *be the set of pointed matroids (M,p) on $S \cup p$ such that p is not a loop of M. There are one-to-one correspondences between* $\mathrm{Semimat}(S)$, $\mathrm{MatId}(S)$, $\mathrm{MatPreim}(S)$, $\mathrm{MatCoext}(S,p)$ *and* $\mathrm{Pointedmat}(S,p)$.

Proof. It suffices to show a correspondence between $\mathrm{MatCoext}(S,p)$ and $\mathrm{Pointedmat}(S,p)$. The elements of $\mathrm{MatCoext}(S,p)$ are the pairs $(\widetilde{N}/p, \widetilde{N})$ for all matroids $\widetilde{N}$ on $S \cup p$ such that $r(\widetilde{N}) > r(\widetilde{N}/p)$; *i.e.*, such that p is not a loop. The map $(\widetilde{N}/p, \widetilde{N}) \mapsto (\widetilde{N}, p)$ establishes the desired bijection. □

At this point, given a set S and an element $p \notin S$, we have bijections between $\mathrm{Semimat}(S)$, $\mathrm{MatId}(S)$, $\mathrm{MatPreim}(S)$, $\mathrm{MatCoext}(S,p)$ and $\mathrm{Pointedmat}(S,p)$, provided by Theorems 3.3.6, 3.4.6, 3.4.9, and 3.5.3. The bijection $\mathrm{Pointedmat}(S,p) \to \mathrm{Semimat}(S)$ is an important one. We have obtained it as the composition of four bijections, and now we wish to describe it explicitly.

Theorem 3.5.4 *Let S be a set and let $p \notin S$.*

1. *For $(\widetilde{N}, p) \in \mathrm{Pointedmat}(S,p)$, let $\mathcal{C} = \{A \subseteq S \mid p \notin \mathrm{cl}_{\widetilde{N}}(A)\}$ and let $r_{\mathcal{C}}$ be the restriction of $r_{\widetilde{N}}$ to $\mathcal{C}$. Then $(S, \mathcal{C}, r_{\mathcal{C}})$ is a semimatroid.*

2. *For $(S, \mathcal{C}, r_{\mathcal{C}}) \in \mathrm{Semimat}(S)$, define $r_{\widetilde{N}} : 2^{S \cup p} \to \mathbb{N}$ by*

$$r_{\widetilde{N}}(A) = \begin{cases} r_{\mathcal{C}}(A) & \text{if } A \in \mathcal{C} \\ \max\{r_{\mathcal{C}}(B) \mid B \subseteq A, B \in \mathcal{C}\} + 1 & \text{if } A \notin \mathcal{C} \end{cases}$$

$$r_{\widetilde{N}}(A \cup p) = \begin{cases} r_{\widetilde{N}}(A) + 1 & \text{if } A \in \mathcal{C} \\ r_{\widetilde{N}}(A) & \text{if } A \notin \mathcal{C} \end{cases}$$

   for $A \subseteq S$. Then $r_{\widetilde{N}}$ is a rank function on $S \cup p$, and $(\widetilde{N}, p) \in \mathrm{Pointedmat}(S,p)$.

3. *The two maps* $\mathrm{Pointedmat}(S,p) \to \mathrm{Semimat}(S)$ *and* $\mathrm{Semimat}(S) \to \mathrm{Pointedmat}(S,p)$ *defined above are inverses. They establish a one-to-one correspondence between* $\mathrm{Pointedmat}(S,p)$ *and* $\mathrm{Semimat}(S)$.

Proof. We will show that, if we start with $(\widetilde{N}, p) \in \text{Pointedmat}(S, p)$ and trace the bijections of Theorems 3.5.3, 3.4.8, 3.4.5 and 3.3.6, we obtain the semimatroid $\mathcal{C}(\widetilde{N}, p)$.

Under the bijection of Theorem 3.5.3, $(\widetilde{N}, p) \in \text{Pointedmat}(S, p)$ corresponds to $(\widetilde{N}/p, \widetilde{N}) \in \text{MatCoext}(S, p)$.

Under the bijection of Theorem 3.4.8, $\widetilde{N} \in \text{Coext}(\widetilde{N}/p)$ corresponds to $\widetilde{N} - p \in \text{Preim}(\widetilde{N}/p)$.

$\widetilde{N} - p \in \text{Preim}(\widetilde{N}/p)$, under the bijection of Theorem 3.4.5, corresponds to the modular ideal $\mathcal{C} = \{A \subseteq S \,|\, r_{\widetilde{N}/p}(A) = r_{\widetilde{N}-p}(A)\} \in \text{Ideal}(\widetilde{N}/p)$. Since p is not a loop of $\widetilde{N}$, $r_{\widetilde{N}/p}(A) = r_{\widetilde{N}}(A \cup p) - 1$ and $r_{\widetilde{N}-p}(A) = r_{\widetilde{N}}(A)$. Therefore $\mathcal{C} = \{A \subseteq S \,|\, p \notin \text{cl}_{\widetilde{N}}(A)\}$.

Finally, under the bijection of Theorem 3.3.6, $(\widetilde{N}/p, \mathcal{C}) \in \text{MatId}(S)$ corresponds to $(S, \mathcal{C}, r_{\mathcal{C}}) \in \text{Semimat}(S)$.

Similarly, if we start with a semimatroid $(S, \mathcal{C}, r_{\mathcal{C}})$ and keep track of its successive images under the bijections of Theorems 3.3.6, 3.4.5, 3.4.8 and 3.5.3, we get the pointed matroid $(\widetilde{N}, p)$ described.

This theorem then becomes a consequence of the previous ones. $\square$

It is not difficult to see that, under the coning construction, the central subsets of a hyperplane arrangement $\mathcal{A}$ correspond to the subsets of $c\mathcal{A}$ whose closure in $M_{c\mathcal{A}}$ does not contain the additional hyperplane H. Theorem 3.5.4 shows that, for semimatroids, the natural analog of the cone of a semimatroid $\mathcal{C}$ is the matroid $\widetilde{N}$ of the pointed matroid $(\widetilde{N}, p)$ corresponding to it.

The triple of matroids $(\widetilde{N}, \widetilde{N} - p, \widetilde{N}/p) = (\widetilde{N}, N, M)$ is sometimes called the *triple of the pointed matroid* $(\widetilde{N}, p)$. We will also call it the *triple of the semimatroid* $\mathcal{C}$.

3.6 Geometric semilattices

We now discuss geometric semilattices and their relationship to semimatroids. We start by recalling some poset terminology. For more background information, see for example [56, Chapter 3].

A *meet semilattice* is a poset K such that any subset $S \subseteq K$ has a *greatest lower bound* or *meet* $\wedge S$: an element such that $\wedge S \leq s$ for all $s \in S$, and $\wedge S \geq t$ for any $t \in K$ such that $t \leq s$ for all $s \in S$. Such posets have a minimum element $\hat{0}$.

Notice that if a set S of elements of a meet semilattice has an upper bound, then it has a least upper bound, or *join* $\vee S$. It is the meet of the upper bounds of S.

A *lattice* is a poset L such that any subset $S \subseteq L$ has a greatest lower bound and a least upper bound. Clearly, if a meet semilattice has a maximum element, then it is a lattice.

A meet semilattice K is *ranked* with *rank function* $r : K \to \mathbb{N}$ if, for all $x \in K$, every maximal chain from $\hat{0}$ to x has the same length $r(x)$. An *atom* is an element of rank 1. A set of atoms A is *independent* if it has an upper bound and $r(\vee A) = |A|$.

Definition 3.6.1 *A* geometric semilattice *is a ranked meet semilattice satisfying the following two conditions.*

(G1) *Every element is a join of atoms.*

(G2) *The collection of independent set of atoms is the collection of independent sets of a matroid.*

A geometric lattice *is a ranked lattice satisfying (G1) and (G2).*

Geometric lattices arise very naturally in matroid theory from the following result. Recall that a matroid $M = (S, r)$ is *simple* if $r(x) = 1$ for all $x \in S$ and $r(\{x, y\}) = 2$ for all $x, y \in S, x \neq y$.

Theorem 3.6.2 *[9], [21] A poset is a geometric lattice if and only if it is isomorphic to the poset of flats of a matroid. Furthermore, each geometric lattice is the poset of flats of a unique simple matroid, up to isomorphism.*

From this point of view, semimatroids are the "right" generalization of matroids, as the following theorem shows.

Definition 3.6.3 *A semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ is* simple *if $\{x\} \in \mathcal{C}$ and $r_{\mathcal{C}}(x) = 1$ for all $x \in S$, and $r_{\mathcal{C}}(\{x, y\}) = 2$ for all $\{x, y\} \in \mathcal{C}$ with $x \neq y$.*

Theorem 3.6.4 *A poset is a geometric semilattice if and only if it is isomorphic to the poset of flats of a semimatroid. Furthermore, each geometric semilattice is the poset of flats of a unique simple semimatroid, up to isomorphism.*

To prove Theorem 3.6.4 we use the following two propositions.

Proposition 3.6.5 *[66] A poset K is a geometric semilattice if and only if there is a geometric lattice L with an atom p such that $K = L - [p, \hat{1}]$.* [1] *Furthermore, L and p are uniquely determined by K.*

Proposition 3.6.6 *Let $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ be a semimatroid, and let $(\widetilde{N}, p)$ be the pointed matroid on $S \cup p$ corresponding to it under the bijection of Theorem 3.5.4. Let $K(\mathcal{C})$ and $L(\widetilde{N})$ be the posets of flats of $\mathcal{C}$ and $\widetilde{N}$. Then $K(\mathcal{C}) = L(\widetilde{N}) - [p, \hat{1}]$.*

Proof. Since both posets are ordered by containment, we only need to show the equality of the sets $K(\mathcal{C})$ and $L(\widetilde{N}) - [p, \hat{1}]$.

First we show that $K(\mathcal{C}) \subseteq L(\widetilde{N}) - [p, \hat{1}]$. Let $X \in K(\mathcal{C})$. Then for all $x \notin X$ such that $X \cup x \in \mathcal{C}$, $r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X) + 1$, and therefore $r_{\widetilde{N}}(X \cup x) = r_{\widetilde{N}}(X) + 1$. To check that X is a flat in $\widetilde{N}$, we need to show that this equality still holds if $X \cup x \notin \mathcal{C}$. This is not difficult: if that is the case and $x \neq p$, then $r_{\widetilde{N}}(X \cup x) = \max\{r_{\mathcal{C}}(Y) \mid Y \subseteq X \cup x, Y \in \mathcal{C}\} + 1 \geq r_{\mathcal{C}}(X) + 1 = r_{\widetilde{N}}(X) + 1$. Clearly then equality must hold. The case $x = p$ is easier, but needs to be checked separately.

Hence $K(\mathcal{C}) \subseteq L(\widetilde{N})$, and since no element of $\mathcal{C}$ contains p, $K(\mathcal{C}) \subseteq L(\widetilde{N}) - [p, \hat{1}]$.

The inverse inclusion is easier. If X is a flat in $\widetilde{N}$ not containing p, then $r_{\widetilde{N}}(X \cup x) = r_{\widetilde{N}}(X) + 1$ for all $x \notin X$. When $X \cup x \in \mathcal{C}$, this equality says that $r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X) + 1$. Therefore X is a flat in $\mathcal{C}$ also. □

Proof of Theorem 3.6.4. It is not difficult to check that the bijection of Theorem 3.5.4 restricts to a bijection between *simple pointed matroids* (pointed matroids $(\widetilde{N}, p) \in \text{Pointedmat}(S, p)$ such that $\widetilde{N}$ is simple) and simple semimatroids. The result then follows combining this fact with Theorem 3.6.2 and Propositions 3.6.5 and 3.6.6. □

[1] Here $[p, \hat{1}]$ denotes the interval of elements greater than or equal to p in the poset L.

3.7 Duality, deletion and contraction

Like matroids, semimatroids have natural notions of duality, deletion and contraction, which we now define.

Definition 3.7.1 *Let $\mathcal{C} = (S, \mathcal{C}, r_\mathcal{C})$ be a semimatroid. Extend the function $r_\mathcal{C}$ to a matroid rank function $r : 2^S \to \mathbb{N}$ as in Proposition 3.3.1. Define the simplicial complex $\mathcal{C}^* = \{X \subseteq S \mid S - X \notin \mathcal{C}\}$, and the rank function $r^* : \mathcal{C}^* \to \mathbb{N}$ by $r^*(X) = |X| - r + r(S - X)$. The* dual of $\mathcal{C}$ *is the triple $\mathcal{C}^* = (S, \mathcal{C}^*, r^*)$.*

Proposition 3.7.2 *$\mathcal{C}^*$ is a semimatroid.*

Proof. It is possible to simply check that $\mathcal{C}^*$ satisfies the axioms of a semimatroid. It is shorter to proceed as follows.

Consider the pair $(M, N) \in \mathrm{MatPreim}(S)$ associated to $\mathcal{C}$ under Corollary 3.4.6. It is known [39, Proposition 8.1.6(f)] that if N is an elementary preimage of M, then M^* is an elementary preimage of N^*. From the pair $(N^*, M^*) \in \mathrm{MatPreim}(S)$, we then get a semimatroid using Corollary 3.4.6 again. It is straightforward to check that this semimatroid is precisely $\mathcal{C}^*$. □

Proposition 3.7.3 *For any semimatroid $\mathcal{C}$, we have that $(\mathcal{C}^*)^* = \mathcal{C}$.*

Proof. This is easy to check directly from the definition. □

Definition 3.7.4 *Let $\mathcal{C} = (S, \mathcal{C}, r_\mathcal{C})$ be a semimatroid and let $e \in S$ be such that $\{e\} \in \mathcal{C}$. Let $\mathcal{C}/e = \{A \subseteq S - e \mid A \cup e \in \mathcal{C}\}$ and, for $A \in \mathcal{C}/e$, let $r_{\mathcal{C}/e}(A) = r_\mathcal{C}(A \cup e) - r_\mathcal{C}(e)$. The* contraction of e from $\mathcal{C}$ *is the triple $\mathcal{C}/e = (S - e, \mathcal{C}/e, r_{\mathcal{C}/e})$.*

Proposition 3.7.5 *$\mathcal{C}/e$ is a semimatroid.*

Proof. Checking the axioms of a semimatroid is straightforward. □

Definition 3.7.6 *Let $\mathcal{C} = (S, \mathcal{C}, r_\mathcal{C})$ be a semimatroid and let $e \in S$ be such that $\{e\} \in \mathcal{C}$. Let $\mathcal{C} - e = \{A \in \mathcal{C} \mid e \notin A\}$ and, for $A \in \mathcal{C} - e$, let $r_{\mathcal{C}-e}(A) = r_\mathcal{C}(A)$. The* deletion of e from $\mathcal{C}$ *is the triple $\mathcal{C} - e = (S - e, \mathcal{C} - e, r_{\mathcal{C}-e})$.*

Proposition 3.7.7 *$\mathcal{C} - e$ is a semimatroid.*

Proof. Checking the axioms of a semimatroid is straightforward. □

Again, as with matroids, there are two special kinds of elements that we need to pay special attention to when we perform deletion and contraction.

Definition 3.7.8 *A* loop *of a semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ is an element $e \in S$ such that $\{e\} \in \mathcal{C}$ and $r_{\mathcal{C}}(e) = 0$.*

Definition 3.7.9 *An* isthmus *of a semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ is an element $e \in S$ such that, for all $A \in \mathcal{C}$, $A \cup e \in \mathcal{C}$ and $r_{\mathcal{C}}(A \cup e) = r_{\mathcal{C}}(A) + 1$.*

Lemma 3.7.10 *If $e \in S$ is a loop of the semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$, then $r_{\mathcal{C}/e} = r_{\mathcal{C}}$. Otherwise, $r_{\mathcal{C}/e} = r_{\mathcal{C}} - 1$.*

Proof. Clearly $r_{\mathcal{C}/e} \leq r_{\mathcal{C}}$. If e is a loop, consider any $A \in \mathcal{C}$. (CR1') applies to $\{e\}$ and A, so $A \cup e \in \mathcal{C}$ and $r_{\mathcal{C}}(A \cup e) = r_{\mathcal{C}}(A)$. Therefore the maximum rank $r_{\mathcal{C}}$ in $\mathcal{C}$ is achieved for some $A \in \mathcal{C}/e$. But then we have $r_{\mathcal{C}/e}(A) = r_{\mathcal{C}}(A \cup e) - 0 = r_{\mathcal{C}}$, so $r_{\mathcal{C}/e} = r_{\mathcal{C}}$.

If e is not a loop, then for all $A \in \mathcal{C}/e$ we have $r_{\mathcal{C}/e}(A) = r_{\mathcal{C}}(A \cup e) - 1$, so $r_{C/e} \leq r_C - 1$. Equality holds: if we start with $\{e\} \in \mathcal{C}$ and repeatedly apply (CR2') with an element of $\mathcal{C}$ of rank $r_{\mathcal{C}}$, we can obtain a set $A \cup e$ of rank $r_{\mathcal{C}}$. Then $r_{C/e}(A) = r_C - 1$. □

Lemma 3.7.11 *If $e \in S$ is an isthmus of the semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$, then $r_{\mathcal{C}-e} = r_{\mathcal{C}} - 1$. Otherwise, $r_{\mathcal{C}-e} = r_{\mathcal{C}}$.*

Proof. Clearly $r_{\mathcal{C}-e} \leq r_{\mathcal{C}}$. If e is an isthmus then it is clear from the definition that $r_{\mathcal{C}-e} = r_{\mathcal{C}} - 1$.

If e is not an isthmus, there are two cases. If there is an $A \in \mathcal{C}$ such that $A \cup e \notin \mathcal{C}$, take a maximal one. It is also a maximal set in $\mathcal{C}$, so it has maximum rank $r_{\mathcal{C}}$; and $A \in \mathcal{C} - e$, so $r_{\mathcal{C}-e} = r_{\mathcal{C}}$. The other possibility is that for all $A \in \mathcal{C}$, we have $A \cup e \in \mathcal{C}$ and $r(A \cup e) = r(A)$. In this case it is also clear that $r_{\mathcal{C}-e} = r_{\mathcal{C}}$. □

Lemma 3.7.12 *If $e \in S$ is a loop or an isthmus of the semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$, then $\mathcal{C} - e = \mathcal{C}/e$.*

Proof. This is clear from Lemmas 3.7.10 and 3.7.11 and their proofs. □

3.8 The Tutte polynomial

With the background results that we have established, we are now able to define and study the Tutte polynomial of a semimatroid.

Definition 3.8.1 *The* Tutte polynomial *of a semimatroid $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ is defined by*

$$T_{\mathcal{C}}(x, y) = \sum_{X \in \mathcal{C}} (x-1)^{r_{\mathcal{C}} - r_{\mathcal{C}}(X)} (y-1)^{|X| - r_{\mathcal{C}}(X)}. \tag{3.8.1}$$

Proposition 3.8.2 *Let $\mathcal{A}$ be a hyperplane arrangement and let $\mathcal{C}_{\mathcal{A}}$ be the semimatroid determined by it. Then $T_{\mathcal{A}}(x, y) = T_{\mathcal{C}_{\mathcal{A}}}(x, y)$.*

Proof. This is clear from the definitions. □

Example. Figure 3-2 shows a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^3$, consisting of the five planes $x + y + z = 0, x = y, y = z, z = x$ and $x + y + z = 1$ in that order.

Table 3.1 shows all the central subsets of $\mathcal{A}$, and their contributions to the Tutte polynomial of $\mathcal{A}$. We find that

$$\begin{aligned} T_{\mathcal{A}}(x, y) &= (x-1)^3 + 5(x-1)^2 + 9(x-1) + 6 + (x-1)(y-1) + 2(y-1) \\ &= x^3 + 2x^2 + xy + x + y. \end{aligned}$$

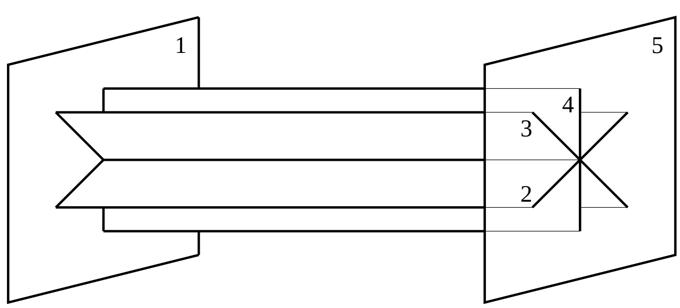

Figure 3-2: The arrangement $\mathcal{A}$.

central subset of $\mathcal{A}$	contribution to $T_{\mathcal{A}}(x,y)$
$\emptyset$	$(x-1)^3(y-1)^0$
$1,2,3,4,5$	$(x-1)^2(y-1)^0$
$12,13,14,23,24,25,34,35,45$	$(x-1)^1(y-1)^0$
$123,124,134,235,245,345$	$(x-1)^0(y-1)^0$
234	$(x-1)^1(y-1)^1$
$1234,2345$	$(x-1)^0(y-1)^1$

Table 3.1: Computing the Tutte polynomial $T_{\mathcal{A}}(x,y)$.

As with matroids, the Tutte polynomial satisfies the following nice recursion, known as the *deletion-contraction* relation.

Proposition 3.8.3 *Let $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ be a semimatroid, and let $e \in S$ be such that $\{e\} \in \mathcal{C}$.*

(i) $T_{\mathcal{C}}(x,y) = T_{\mathcal{C}-e}(x,y) + T_{\mathcal{C}/e}(x,y)$ if e is neither an isthmus nor a loop and $\{e\} \in \mathcal{C}$.

(ii) $T_{\mathcal{C}}(x,y) = x\,T_{\mathcal{C}-e}(x,y)$ if e is an isthmus.

(iii) $T_{\mathcal{C}}(x,y) = y\,T_{\mathcal{C}/e}(x,y)$ if e is a loop.

(iv) If $e \in S$ and $\{e\} \notin \mathcal{C}$ then $T_{(S,\mathcal{C},r_{\mathcal{C}})}(x,y) = T_{(S-e,\mathcal{C},r_{\mathcal{C}})}(x,y)$.

Proof. We have

$$T_{\mathcal{C}}(x,y) = \sum_{\substack{X \in \mathcal{C} \\ e \notin X}} (x-1)^{r_{\mathcal{C}} - r_{\mathcal{C}}(X)} (y-1)^{|X| - r_{\mathcal{C}}(X)} + \sum_{X \cup e \in \mathcal{C}} (x-1)^{r_{\mathcal{C}} - r_{\mathcal{C}}(X\cup e)} (y-1)^{|X \cup e| - r_{\mathcal{C}}(X \cup e)}.$$

Notice that, if $r_{\mathcal{C}} = r_{\mathcal{C}-e}$, the first sum in the right hand side is exactly the Tutte polynomial of $\mathcal{C} - e$. If, on the other hand, $r_{\mathcal{C}} = r_{\mathcal{C}-e} + 1$, the only difference is that we get an extra factor of $(x-1)$. More precisely, in view of Lemma 3.7.11, the first sum of the right hand side is $T_{\mathcal{C}-e}(x,y)$ if e is not an isthmus, and $(x-1)T_{\mathcal{C}-e}(x,y)$ if it is an isthmus.

Similarly, from Lemma 3.7.10, the second sum is $T_{\mathcal{C}/e}(x, y)$ if e is not a loop, and $(y-1)T_{\mathcal{C}/e}(x, y)$ if it is a loop.

These two observations, together with Lemma 3.7.12, complete the proof of *(i)*-*(iii)*. Also, *(iv)* is clear from the definition. □

Definition 3.8.4 *Two matroids* $(S_1, \mathcal{C}_1, r_{\mathcal{C}_1})$ *and* $(S_2, \mathcal{C}_2, r_{\mathcal{C}_2})$ *are* isomorphic *if there is a bijection* $f : S_1 \to S_2$ *which induces an isomorphism of simplicial complexes* $f : \mathcal{C}_1 \to \mathcal{C}_2$ *such that* $r_{\mathcal{C}_1}(c) = r_{\mathcal{C}_2}(f(c))$ *for all* $c \in \mathcal{C}_1$.

A function f on the class $\mathbb{S}$ of semimatroids is called a *semimatroid invariant* if $f(\mathcal{C}_1) = f(\mathcal{C}_2)$ for all $\mathcal{C}_1 \cong \mathcal{C}_2$. An invariant is called a *Tutte-Grothendieck invariant* (or *T-G invariant*) if it satisfies the conditions of Proposition 3.8.3. The following theorem shows that the Tutte polynomial is not only a T-G invariant; in fact it is the universal T-G invariant on the class of semimatroids. Any other *generalized T-G invariant*, that is, an invariant satisfying the conditions of Theorem 3.8.6, is an evaluation of the Tutte polynomial. An equivalent result is essentially known for matroids [15], [45].

Definition 3.8.5 *For a semimatroid* $\mathcal{C} = (S, \mathcal{C}, r)$*, let* $\#\mathcal{C}$ *be the number of elements* $x \in S$ *such that* $\{x\} \in \mathcal{C}$*. A semimatroid is* non-trivial *if* $\#\mathcal{C} \neq 0$.

Theorem 3.8.6 *Let* $\mathbb{S}$ *be the class of non-trivial semimatroids. Let* $\Bbbk$ *be a field and* $a, b \in \Bbbk$*; and let* R *be a commutative ring containing* $\Bbbk$*. Let* $f : \mathbb{S} \to R$ *be a* generalized T-G invariant*; i.e.,*

(i) *if* $\mathcal{C}_1 \cong \mathcal{C}_2$ *then* $f(\mathcal{C}_1) = f(\mathcal{C}_2)$.

(ii) *If* $e \in S$ *is neither an isthmus nor a loop in* $\mathcal{C} = (S, \mathcal{C}, r_{\mathcal{C}})$ *and* $\{e\} \in \mathcal{C}$*, then* $f(\mathcal{C}) = af(\mathcal{C} - e) + bf(\mathcal{C}/e)$.

(iii) *If* e *is an isthmus in* $\mathcal{C}$*, then* $f(\mathcal{C}) = f(I)f(\mathcal{C} - e)$.

(iv) *If* e *is a loop in* $\mathcal{C}$*, then* $f(\mathcal{C}) = f(L)f(\mathcal{C}/e)$.

(v) *If* $e \in S$ *and* $\{e\} \notin \mathcal{C}$ *then* $f(S, \mathcal{C}, r_{\mathcal{C}}) = f(S - e, \mathcal{C}, r_{\mathcal{C}})$.

Then the function f is given by $f(\mathcal{C}) = a^{\#\mathcal{C}-r_\mathcal{C}}\, b^{r_\mathcal{C}}\, T_\mathcal{C}(f(I)/b, f(L)/a)$ for $\mathcal{C} = (S, \mathcal{C}, r_\mathcal{C})$.

Here $I = (\{i\}, \{\emptyset, \{i\}\}, r)$ denotes the semimatroid consisting of a single isthmus i, and $L(\{l\}, \{\emptyset, \{l\}\}, r)$ denotes the semimatroid consisting of a single loop l.

Proof. We can proceed by induction. The only non-trivial semimatroids which cannot be decomposed using $(ii), (iii), (iv)$ and (v) are I and L, in which case the formula for $f(\mathcal{C})$ holds trivially. It simply remains to show that $a^{\#\mathcal{C}-r_\mathcal{C}}\, b^{r_\mathcal{C}} T_\mathcal{C}(f(I)/b, f(L)/a)$ satisfies the relations $(ii), (iii), (iv)$ and (v). This is straightforward from Proposition 3.8.3. □

We conclude this section with some remarks about the relationship between the Tutte polynomial of a semimatroid $\mathcal{C}$, the Tutte polynomials of its associated triple $(\widetilde{N}, N, M)$, and the Tutte polynomial of the dual semimatroid $\mathcal{C}^*$.

In the study of the characteristic polynomial $\chi(q)$ of an affine hyperplane arrangement, the coning construction of Definition 3.5.2 is fundamental, due to the following result.

Proposition 3.8.7 ([44, Proposition 2.51]) *For any arrangement $\mathcal{A}$,*

$$\chi_{c\mathcal{A}}(q) = (q-1)\chi_\mathcal{A}(q).$$

This proposition tells us that, to study characteristic polynomials of arrangements, we can essentially focus our attention on central arrangements. Proposition 3.8.7 generalizes immediately to semimatroids.

As we saw in Theorem 3.5.4, the analog of the cone of an arrangement $\mathcal{A}$ is the matroid $\widetilde{N}$ of the semimatroid $\mathcal{C}$. If, in analogy with the definition for arrangements, we define the *characteristic polynomial of the semimatroid $\mathcal{C}$* to be $\chi_\mathcal{C}(q) = (-1)^r T_\mathcal{C}(1-q, 0)$, we have the following proposition.

Proposition 3.8.8 *For any semimatroid $\mathcal{C}$,*

$$\chi_{\widetilde{N}}(q) = (q-1)\chi_\mathcal{C}(q).$$

We might wonder if this result generalizes to the Tutte polynomial. It turns out that this situation is not so simple. Let

$$U_{\mathcal{C}}(x,y) = \sum_{X \notin \mathcal{C}} (x-1)^{r_M - r_M(X)} (y-1)^{|X| - r_M(X)}. \tag{3.8.2}$$

Then, by looking at the defining sums of T_M, T_N and $T_{\widetilde{N}}$, it is easy to see that $T_M = T_{\mathcal{C}} + U_{\mathcal{C}}$, $T_N = (x-1)T_{\mathcal{C}} + U_{\mathcal{C}}/(y-1)$, and $T_{\widetilde{N}} = xT_{\mathcal{C}} + y/(y-1)U_{\mathcal{C}}$. (The third of these equations proves Proposition 3.8.8.) This means that we *can* express the Tutte polynomial of $\mathcal{C}$ in terms of the Tutte polynomials of these three matroids M, N and $\widetilde{N}$, by solving for $T_{\mathcal{C}}$ in any two of these three equations. However, $T_{\mathcal{C}}$ does not only depend on $T_{\widetilde{N}}$. This simple dependence takes place for the characteristic polynomial only because the second term in the expression of $T_{\widetilde{N}}$ vanishes when we substitute $x = 1-q$ and $y = 0$.

We conclude that the Tutte polynomial of a semimatroid is closely related to the Tutte polynomials of its associated triple $(\widetilde{N}, N, M)$. However, the relationship is not simple enough that we can derive our results on Tutte polynomials of semimatroids as simple consequences of the analogous results for matroids.

Now let us discuss duality and the Tutte polynomial. For matroids M, we know that $T_{M^*}(x,y) = T_M(y,x)$. This is not the case for a semimatroid $\mathcal{C}$. In fact, it is not difficult to see that $T_{\mathcal{C}^*}(x,y) = U_{\mathcal{C}}(y,x)/(x-1)$.

It is possible to define a three-variable Tutte-like polynomial of a semimatroid which is more compatible with duality. The construction is essentially the same as one by Las Vergnas [41], who defined the concept of the *Tutte polynomial of a quotient map.* In fact, if the semimatroid $\mathcal{C}$ corresponds to the quotient map $N \to M$ under Corollary 3.4.6, then our definition of the Tutte polynomial of $\mathcal{C}$ coincides with the coefficient of z in Las Vergnas's definition of the Tutte polynomial of the quotient map $N \to M$.

3.9 Basis activity

We now show that the Tutte polynomial of a semimatroid has nonnegative coefficients, by giving a combinatorial interpretation of them. Crapo showed that the coefficients of the Tutte polynomial of a matroid count the bases with a given internal and external activity [20]. Our interpretation in the case of semimatroids is analogous. There are some subtleties involved in extending this result to semimatroids, so we will need to give slightly different definitions of internal and external activity. Our proof will be slightly different from his as well.

In this section we will work with a fixed semimatroid $\mathcal{C} = (S, \mathcal{C}, r)$. We will denote elements of S by lower case letters, and subsets of S by upper case letters. As mentioned after Definition 3.2.1, we will sometimes call the sets in $\mathcal{C}$ central sets. Proposition 3.3.1 shows that the rank function r extends to a matroid rank function on 2^S, which we will also call r. No confusion arises from this notation because the semimatroid and matroid rank functions have the same value where they are both defined.

A *basis* of $\mathcal{C} = (S, \mathcal{C}, r)$ is a set $B \in \mathcal{C}$ such that $|B| = r(B) = r$. A set $X \in \mathcal{C}$ is *dependent* if $r(X) < |X|$ and *independent* otherwise. A *circuit* C of $\mathcal{C}$ is a minimal dependent set in $\mathcal{C}$. Clearly such a set satisfies $r(C) = |C| - 1$. A *bond* D is a minimal subset of S whose deletion from $\mathcal{C}$ makes the rank of $\mathcal{C}$ decrease; *i.e.*, one such that $r(S - D) < r$, where $r = r(S)$ is the rank of $\mathcal{C}$. Clearly a bond satisfies $r(S - D) = r - 1$.

Lemma 3.9.1 *Let B be a basis of $\mathcal{C}$, and let $e \notin B$ be such that $B \cup e \in \mathcal{C}$. Then $B \cup e$ contains a unique circuit.*

Proof. Since $B \cup e \in \mathcal{C}$ is dependent, it contains a circuit. Now assume that it contains two different circuits C_1 and C_2. By (R3) we know that

$$\begin{aligned} r(C_1 \cap C_2) + r(C_1 \cup C_2) &\leq r(C_1) + r(C_2) \\ &= |C_1| - 1 + |C_2| - 1 \\ &= |C_1 \cap C_2| - 1 + |C_1 \cup C_2| - 1. \end{aligned}$$

But $r(B \cup e) = |B \cup e| - 1$ so, by (R2'), $r(X) \geq |X| - 1$ for all $X \subseteq B \cup e$. Therefore $r(C_1 \cap C_2) = |C_1 \cap C_2| - 1$ and $r(C_1 \cup C_2) = |C_1 \cup C_2| - 1$. Thus $C_1 \cap C_2$ is a dependent set in $\mathcal{C}$, and it is a proper subset of the circuit C_1. This is a contradiction. □

Lemma 3.9.2 *Let B be a basis of $\mathcal{C}$, and let $i \in B$. Then $S - B \cup i$ contains a unique bond.*

Proof. The deletion of $S - B \cup i$ from $\mathcal{C}$ makes the rank of $\mathcal{C}$ decrease, so this set contains a bond. Assume that it contains two different bonds B_1 and B_2. Then

$$\begin{aligned} r(S - (B_1 \cap B_2)) &= r((S - B_1) \cup (S - B_2)) \\ &\leq r(S - B_1) + r(S - B_2) - r((S - B_1) \cap (S - B_2)) \\ &= (r - 1) + (r - 1) - r(S - (B_1 \cup B_2)). \end{aligned}$$

But $S - (B_1 \cup B_2) \supseteq B - i$ and $r(B - i) = r - 1$, so $r(S - (B_1 \cup B_2)) \geq r - 1$. It follows that $r(S - (B_1 \cap B_2)) \leq r - 1$. Hence the removal of $B_1 \cap B_2$ makes the rank of the semimatroid decrease, and $B_1 \cap B_2$ is a proper subset of the bond B_1. This is a contradiction. □

From now on, we will fix a linear order on S. Now each k-subset of S corresponds to a strictly increasing sequence of k numbers between 1 and $|S|$. For each $0 \leq k \leq |S|$, order the k-subsets of S using the lexicographic order on these sequences.

Definition 3.9.3 *Let B be a basis of $\mathcal{C}$. An element $e \notin B$ is an* externally active element *for B if $B \cup e \in \mathcal{C}$ and e is the smallest element*[2] *of the unique circuit in $B \cup e$. Let $E(B)$ be the set of externally active elements for B, and let $e(B) = |E(B)|$. We call $e(B)$ the* external activity *of B.*

Definition 3.9.4 *Let B be a basis of $\mathcal{C}$. An element $i \in B$ is an* internally active element *in B if i is the smallest element of the unique bond in $S - B \cup i$. Let $I(B)$ be the set of internally active elements for B, and let $i(B) = |I(B)|$. We call $i(B)$ the* internal activity *of B.*

[2] according to the fixed linear order

Now we are in a position to state the main theorem of this section.

Theorem 3.9.5 *For any semimatroid $\mathcal{C}$,*

$$T_{\mathcal{C}}(q,t) = \sum_{B \text{ basis of } \mathcal{C}} q^{i(B)} t^{e(B)}.$$

Theorem 3.9.5 shows that the coefficients of the Tutte polynomial are nonnegative integers. The coefficient of $q^i t^e$ is equal to the number of bases of $\mathcal{C}$ with internal activity i and external activity e.

We still have some work to do before we can prove Theorem 3.9.5. The next step will be to give a very useful characterization of internally and externally active elements. From now on, when proving results about internally and externally active elements, we will always use Lemmas 3.9.6 and 3.9.7 instead of the original definitions.

Given $X \subseteq S$ and an element e, let $X_{>e} = \{x \in X \mid x > e\}$. Define $X_{<e}$ analogously.

Lemma 3.9.6 *Let B be a basis of $\mathcal{C}$ and let $e \notin B$ be such that $B \cup e \in \mathcal{C}$. Then e is externally active for B if and only if $r(B_{>e} \cup e) = r(B_{>e})$.*

Proof. First assume that $r(B_{>e} \cup e) = r(B_{>e})$. Then $B_{>e} \cup e \in \mathcal{C}$ is dependent, so it contains a circuit C; e is clearly the smallest element in this circuit. But C must also be the unique circuit contained in $B \cup e$. Therefore e is an externally active element for B.

Now assume that e is externally active for B. The unique circuit in $B \cup e$ obviously contains e; call it $C \cup e$. Then $C \subseteq B_{>e}$. By submodularity, we have $r(B_{>e}) + r(C \cup e) \geq r(B_{>e} \cup e) + r(C)$. But $r(C \cup e) = r(C)$, so $r(B_{>e}) \geq r(B_{>e} \cup e)$ and the desired result follows. □

Lemma 3.9.7 *Let B be a basis and $i \in B$. Then i is internally active in B if and only if $r(B - i \cup S_{<i}) < r$.*[3]

[3] In fact, this is true if and only if $r(B - i \cup S_{<i}) = r - 1$.

Proof. First assume that $r(B - i \cup S_{<i}) < r$. Then the removal of $(S - B)_{>i} \cup i$ makes the rank of the semimatroid drop, so $(S - B)_{>i} \cup i$ contains a bond. This bond must contain i; call it $D \cup i$, where $D \subseteq (S - B)_{>i}$. The smallest element of this bond is i, and this bond must also be the unique bond contained in $S - B \cup i$. Therefore i is an internally active element of B.

Now assume that i is internally active in B. Let $S - D \cup i$ be the unique bond in $S - B \cup i$, where $D \supseteq B$. Since i is the smallest element in this bond, $D \supseteq S_{<i}$. Therefore $B \cup S_{<i} \subseteq D$ and, since $S - D \cup i$ is a bond, $r(B - i \cup S_{<i}) < r(D - i) < r$. □

Now we wish to present a different description of sets in $\mathcal{C}$. To do it, we need two definitions. For each $X \subseteq S$, let dX be the lexicographically largest basis of X. For each independent set X, which is necessarily in $\mathcal{C}$, let uX be the lexicographically smallest basis of $\mathcal{C}$ which contains X.[4] Notice that, for any $X \subseteq S$, udX is a basis of $\mathcal{C}$.

Definition 3.9.8 *Let $\mathcal{T}$ be the set of triples (B, I, E) such that B is a basis of $\mathcal{C}$, $I \subseteq I(B)$ is a set of internally active elements for B, and $E \subseteq E(B)$ is a set of internally active elements of B.*

We will establish a bijection between $\mathcal{T}$ and $\mathcal{C}$. Define two maps ϕ_1 and ϕ_2 as follows. Given $(B, I, E) \in \mathcal{T}$, let $\phi_1(B, I, E) = B - I \cup E$. Given $X \in \mathcal{C}$, let $\phi_2(X) = (udX, udX - dX, X - dX)$. We will show that the maps ϕ_1 and ϕ_2 give the desired bijection: every set $X \in \mathcal{C}$ can be written uniquely in the form $X = B - I \cup E$ where B is a basis of $\mathcal{C}$, $I \subseteq I(B)$ and $E \subseteq E(B)$.

Example. Recall the arrangement $\mathcal{A}$ introduced at the beginning of Section 3.8. Table 3.2 illustrates the bijection between $\mathcal{T}$ and $\mathcal{C}$ in that case. Theorem 3.9.5 and Table 3.2 imply that $T_{\mathcal{A}}(q, t) = q^3 + 2q^2 + qt + q + t$, confirming our computation at the beginning of Section 3.8.

[4]We will extend the definition of uX to all $X \subseteq S$ after the proof of Lemma 3.9.14. For simplicity, we postpone the full definition until then.

B	$I(B)$	$E(B)$	possible $B - I \cup E$
123	123	-	$\emptyset, 1, 2, 3, 12, 13, 23, 123$
124	12	-	$4, 14, 24, 124$
134	1	2	$34, 134, 234, 1234$
235	23	–	$5, 25, 35, 235$
245	2	-	$45, 245$
345	-	2	$345, 2345$

Table 3.2: The bijection between $\mathcal{T}$ and $\mathcal{C}$.

Lemma 3.9.9 *The map ϕ_1 maps $\mathcal{T}$ to $\mathcal{C}$.*

Proof. Let $(B, I, E) \in \mathcal{T}$. For all $e \in E$, $B \cup e$ is central and $r(B \cup e) = r(B)$, so $e \in \text{cl}(B)$. Therefore $E \subseteq \text{cl}(B)$ and $B \cup E \subseteq \text{cl}(B)$. Since $\text{cl}(B) \in \mathcal{C}$, this implies that $B \cup E \in \mathcal{C}$, and $B - I \cup E \in \mathcal{C}$ as well. □

Lemma 3.9.10 *The map ϕ_2 maps $\mathcal{C}$ to $\mathcal{T}$.*

Proof. Let $X \in \mathcal{C}$. Let $d = dX$ and $u = udX$, so that $\phi_2(X) = (u, u - d, X - d)$. We need to show three things.

First, we need u to be a basis for X. This is immediate.

Next, we need the elements of $u - d$ to be internally active in u. Let $x \in u - d$. Since u is the smallest basis for $\mathcal{C}$ containing d, for any element $x' < x$ not in u we have $r(u - x \cup x') = r - 1 = r(u - x)$. By submodularity, we can conclude that $r(u - x \cup S_{<x}) = r - 1$, which is exactly what we wanted.

Finally, we need to show that the elements of $X - d$ are externally active in u. Let $x \in X - d$. First notice that $x \notin u$, because $d \cup x$ is dependent: $r(d \cup x) \leq r(X) = r(d)$. Also notice that $u \cup x$ is central, applying (CR1) to $d \cup x$ and u. Now observe the following. We know that d is the largest basis for X. Therefore $r(d - x' \cup x) = r(d) - 1$ for all $x' \in d_{<x}$. By submodularity, it follows that $r(d - d_{<x} \cup x) = r(d) - |d_{<x}|$. We can rewrite this as $r(d_{>x} \cup x) = r(d_{>x})$ since d is independent. Since $d_{>x} \subseteq u_{>x}$, submodularity implies that $r(u_{>x} \cup x) = r(u_{>x})$. This shows that x is an externally active element in u. □

Proposition 3.9.11 *The map ϕ_1 is a bijection from $\mathcal{T}$ to $\mathcal{C}$, and the map ϕ_2 is its inverse.*

Proposition 3.9.11 is the main ingredient of our proof of Theorem 3.9.5. Before proving it, we need some lemmas.

Lemma 3.9.12 *For all $(B, I, E) \in \mathcal{T}$, we have $r(B - I \cup E) = r - |I|$.*

Proof. We start by showing that $r(B - i \cup e) = r - 1$ for all $i \in I(B), e \in E(B)$. If $e < i$, do the following. Since i is internally active, $r(B - i \cup S_{<i}) = r - 1 = r(B - i)$, and therefore $r(B - i \cup e) = r - 1$. Otherwise, if $i < e$, then $B_{>e} \subseteq B - i$. Since e is externally active, $r(B_{>e} \cup e) = r(B_{>e})$. Submodularity then implies that $r(B - i \cup e) = r(B - i) = r - 1$.

Now that we know this, submodularity implies that $r(B - i \cup E) = r - 1$ for all $i \in I(B), E \subseteq E(B)$. Applying submodularity again, we get $r(B - I \cup E) = r - |I|$ for all $I \subseteq I(B), E \subseteq E(B)$. □

Lemma 3.9.13 *For all $(B, I, E) \in \mathcal{T}$, we have $d(B - I \cup E) = B - I$.*

Proof. Lemma 3.9.12 tells us that $B - I$ is a basis for $B - I \cup E$; we need to show that it is the largest one. Consider an arbitrary $(r - |I|)-$ subset X of $B - I \cup E$ with $X > B - I$. We will show that X is not a basis for $B - I \cup E$.

Let $X = (B - I) - (b_1 \cup \cdots \cup b_k) \cup (e_1 \cup \cdots \cup e_k)$, where the b_i's are in $B - I$ and the e_i's are in E. Since $X > B - I$ we can assume, without loss of generality, that $b_1 < e_1, \ldots, e_k$.

From Lemma 3.9.12 we know that $r(B - I \cup e_i) = r - |I|$ for all $1 \leq i \leq k$. Also, as we saw in the proof of Lemma 3.9.12, having $b_1 \in B$, $e_i \in E(B)$ and $b_1 < e_i$ implies that $r(B - b_1 \cup e_i) = r - 1$. Combining these two inequalities and using submodularity, we get that $r(B - I - b_1 \cup e_i) = r - |I| - 1$ for all $1 \leq i \leq k$. Invoking submodularity once again, we get that $r((B - I) - b_1 \cup (e_1 \cup \cdots \cup e_k)) = r - |I| - 1$. Therefore $r(X) = r((B - I) - (b_1 \cup \cdots \cup b_k) \cup (e_1 \cup \cdots \cup e_k)) \leq r - |I| - 1 < r(B - I \cup E)$. It follows that X is not a basis for $B - I \cup E$. □

Lemma 3.9.14 *For all $(B, I, E) \in \mathcal{T}$, we have $ud(B - I \cup E) = B$.*

Proof. In view of Lemma 3.9.13, we need to show that $u(B - I) = B$. Clearly B is a basis of $\mathcal{C}$ containing $B - I$; now we show that it is the smallest one.

Let $X = B - (b_1 \cup \cdots \cup b_k) \cup (c_1 \cup \cdots \cup c_k)$ be an r-tuple smaller than B, where the b_i's are in I (since X must contain $B - I$) and the c_i's are in S. We will show that X is not a basis for $\mathcal{C}$. Once again we can assume, without loss of generality, that $c_1 < b_1, \ldots, b_k$.

Since each b_i is internally active, $r(B - b_i \cup S_{<b_i}) = r - 1$, and hence $r(B - b_i \cup c_1) = r - 1$. Submodularity gives $r(B - (b_1 \cup \cdots \cup b_k) \cup c_1) = r - k$, which in turn gives $r(X) = r(B - (b_1 \cup \cdots \cup b_k) \cup (c_1 \cup \cdots \cup c_k)) \leq (r - k) + (k - 1) < r$. □

So far we have only defined uX for independent sets X of $\mathcal{C}$. We can extend the definition to arbitrary subsets $X \subseteq S$ as follows. If X is dependent, then there is no basis of $\mathcal{C}$ containing it. Instead, we consider all the minimal sets of rank r which contain X. Let uX be the lexicographically smallest of those sets. Then we can say even more.

Lemma 3.9.15 *For all $(B, I, E) \in \mathcal{T}$, we have $u(B - I \cup E) = B \cup E$ and $du(B - I \cup E) = B$.*

We will not need Lemma 3.9.15 to prove Proposition 3.9.11 and Theorem 3.9.5. We state it for completeness, but we omit its proof, which is very similar to the proofs of Lemmas 3.9.13 and 3.9.14.

Proof of Proposition 3.9.11. Checking that $\phi_1 \circ \phi_2$ is the identity map in $\mathcal{C}$ is immediate, and Lemmas 3.9.13 and 3.9.14 imply that $\phi_2 \circ \phi_1$ is the identity map in $\mathcal{T}$. □

Proof of Theorem 3.9.5. Using the bijection of Proposition 3.9.11, the sets in $\mathcal{C}$ are precisely the sets of the form $B - I \cup E$, where B is a basis, $I \subseteq I(B)$ and $E \subseteq E(B)$.

Also, from Lemma 3.9.12, $r(B - I \cup E) = r - |I|$. Therefore we have

$$\begin{aligned} T(q,t) &= \sum_{X \in \mathcal{C}} (q-1)^{r-r(X)}(t-1)^{|X|-r(X)} \\ &= \sum_{B \text{ basis}} \sum_{I \subseteq I(B)} \sum_{E \subseteq E(B)} (q-1)^{r-r(B-I\cup E)}(t-1)^{|B-I\cup E|-r(B-I\cup E)} \\ &= \sum_{B \text{ basis}} \sum_{I \subseteq I(B)} \sum_{E \subseteq E(B)} (q-1)^{|I|}(t-1)^{|E|} \\ &= \sum_{B \text{ basis}} (1+(q-1))^{|I(B)|}(1+(t-1))^{|E(B)|} \\ &= \sum_{B \text{ basis}} q^{i(B)}t^{e(B)}. \end{aligned}$$

as desired. □

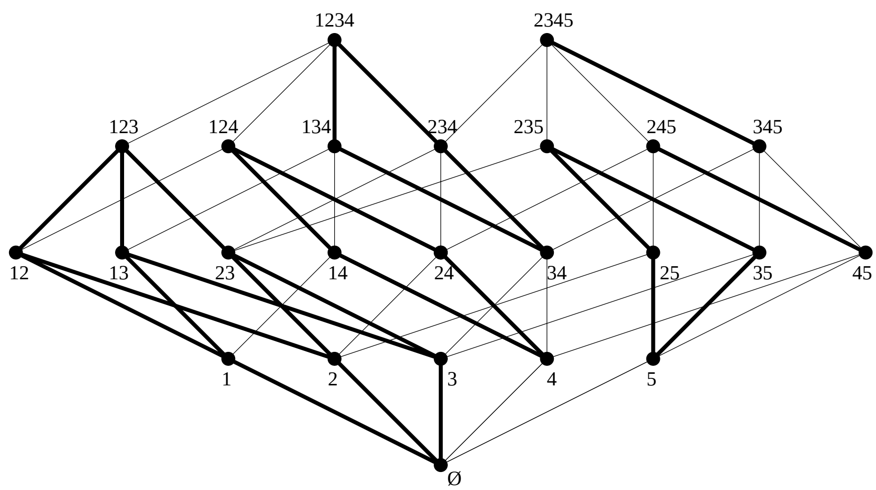

Figure 3-3: The decomposition of $\mathcal{C}$ into intervals.

Regard the simplicial complex $\mathcal{C}$ as a poset, ordering its faces by inclusion. There is a nice way to understand Theorem 3.9.5 in terms of this poset. Proposition 3.9.11 gives us a way of classifying the faces of $\mathcal{C}$ according to the basis of $\mathcal{C}$ that they correspond to under the map ud (or du). This classification decomposes the poset into disjoint intervals, where each interval is a Boolean algebra of the form $[B - I(B), B \cup E(B)]$ for a basis B. This is illustrated in Figure 3-3 for the arrangement $\mathcal{A}$ considered at the beginning of Section 3.8; recall Table 3.2. If we look at the interval corresponding to basis B, and add the contributions of its elements to the right-hand side of (3.8.1), we simply get the monomial $q^{i(B)}t^{e(B)}$.

Chapter 4

An algebra related to the Tutte polynomial

4.1 Introduction

There seems to be a great imbalance between the wide variety of applications and the relatively small existing theory of the Tutte polynomial. In contrast, the characteristic polynomial $\chi(q)$ of a matroid or an arrangement has been understood from many different points of view: the combinatorics of the broken circuit complex, the Orlik-Solomon algebra, the cohomology of the complement of the complexified arrangement, and the Bidigare - Hanlon - Rockmore random walks, for example.

One of the motivations of the work of this chapter is the desire to develop a similar theoretical background for the Tutte polynomial. One would like to be able to study this polynomial from different points of view, perhaps of an algebraic or topological flavor.

Chapter 4 is the beginning of a project which attempts to study the Tutte polynomial from an algebraic point of view. Given a matroid representable over a field of characteristic zero, we construct a graded algebra whose Hilbert-Poincaré series is a simple evaluation of the Tutte polynomial of the matroid.

The work of this chapter is joint work with Alex Postnikov.

4.2 A vector space in $\mathrm{Sym}(E)$

Let $V = \{v_1, \ldots, v_m\}$ be a collection of vectors in an n-dimensional vector space E over a field $\Bbbk$ of characteristic zero. Assume for simplicity that V spans E; if it didn't, we could just restrict our attention to the vector space $\mathrm{span}(V)$.

The set of vectors determines a matroid $M = ([m], r)$. We will use the concept of external activity in M, so we need to fix a linear order on $[m]$: the natural choice is $1 < \cdots < m$. Let bases(M) be the number of bases of M, and bases(M, m) be the number of bases of M with external activity m. Thus $T_M(1, y) = \mathrm{bases}(M, m-n)y^{m-n} + \cdots + \mathrm{bases}(M, 1)y + \mathrm{bases}(M, 0)$ and $T_M(1, 1) = \mathrm{bases}(M)$.

Let $\mathrm{Sym}(E)$ be the symmetric algebra of E. It is a graded algebra $\mathrm{Sym}(E) = \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(E)$, where $\mathrm{Sym}^k(E) = E^{\otimes k}/\mathfrak{b}_k$ and $\mathfrak{b}_k$ is the ideal of $E^{\otimes k}$ generated by all elements of the form $x_1 \otimes \cdots \otimes x_k - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}$, for $x_1, \ldots, x_k \in E$ and $\sigma \in \mathfrak{S}_k$.

For simplicity, if $x, y \in \mathrm{Sym}(E)$, we will sometimes denote $x \otimes y$ in $\mathrm{Sym}(E)$ simply by xy. Also, if $I = \{i_1, \ldots, i_k\} \subseteq [m]$, we will denote the image of $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$ in $\mathrm{Sym}(E)$ simply by v_I, keeping in mind that the order of the v_i's is irrelevant in the product v_I.

Definition 4.2.1 *Let $\mathcal{S}_V$ be the subspace of $\mathrm{Sym}(E)$ generated by the elements v_I for $r([m] - I) = n$. Let $\mathcal{S}_V^k = \mathcal{S}_V \cap \mathrm{Sym}^k(E)$, so $\mathcal{S}_V = \bigoplus_{k=0}^{\infty} \mathcal{S}_V^k$.*

Theorem 4.2.2 *We have $\dim \mathcal{S}_V = \mathrm{bases}(M)$. Furthermore, for $0 \leq k \leq m-n$, we have $\dim \mathcal{S}_V^k = \mathrm{bases}(M, m-n-k)$.*

We start by constructing a spanning set for $\mathcal{S}_V$, which will later turn out to be a basis.

Lemma 4.2.3 *The set $B_V = \{v_{[m]-B-E(B)} \mid B \text{ basis of } M\}$ spans $\mathcal{S}_V$.*

Furthermore, $B_V^k = \{v_{[m]-B-E(B)} \mid B \text{ basis of } M \text{ with } e(B) = m-n-k\}$ spans $\mathcal{S}_V^k$ for $0 \leq k \leq m-n$.

Proof. Since $r(B \cup E(B)) = n$ for any basis B, we have $B_V \subseteq \mathcal{S}_V$

Proceed by contradiction: assume that not all the generators of $\mathcal{S}_V$ are in $\text{span}(B_V)$. Let $[m] - S$ be the lexicographically largest set such that $v_{[m]-S} \notin \text{span}(B_V)$. Then $r(S) = n$, so there exists a basis B of M and a subset $E \subseteq E(B)$ such that $S = B \cup E$, by Lemma 3.9.12. Since $[m] - S \notin B_V$, we have $E \neq E(B)$.

Let $e \in E(B) - E$. Then there is a circuit $C \cup e$ of M with $C \subseteq B$ whose minimal element is e. Thus there exist scalars c_i for $i \in C$ such that $v_e = \sum_{i \in C} c_i v_i$ in E. Since $e \in [m] - S$, we have the following equalities in $\text{Sym}(E)$.

$$\begin{aligned} v_{[m]-S} &= v_{[m]-S-e} \otimes v_e \\ &= v_{[m]-S-e} \otimes \left(\sum_{i \in C} c_i v_i \right) \\ &= \sum_{i \in C} c_i \, v_{[m]-S-e\cup i}. \end{aligned}$$

Now, we know that $e < i$ for all $i \in C$. Therefore $[m] - S - e \cup i$ is lexicographically larger than $[m] - S$, and hence $v_{[m]-S-e\cup i} \in \text{span}(B_V)$ by the maximality of $[m] - S$. It follows that $v_{[m]-S} \in \text{span}(B_V)$ also, a contradiction.

Thus B_V spans $\mathcal{S}_V$, and therefore $B_V^k = B_V \cap \mathcal{S}_V^k$ spans $\mathcal{S}_V^k$ for $0 \leq k \leq m - n$. □

Proof of Theorem 4.2.2. It follows from Lemma 4.2.3 that $\dim \mathcal{S}_V \leq \text{bases}(M)$ and $\dim \mathcal{S}_V^k \leq \text{bases}(M, m - n - k)$. It suffices to prove that $\dim \mathcal{S}_V = \text{bases}(M)$; the second equality will then follow since $\dim \mathcal{S}_V = \sum_k \dim \mathcal{S}_V^k$ and $\text{bases}(M) = \sum_k \text{bases}(M, m - n - k)$.

We prove $\dim \mathcal{S}_V = \text{bases}(M)$ by induction on $|V|$. For $|V| = 1$, the result is trivial. For the induction step, consider V with $|V| \geq 2$. If M only contains loops and coloops, the result is also trivial. So assume, without loss of generality, that m is not a loop or an isthmus in M. The idea now is to use a deletion-contraction argument. The set of vectors $V - v_m$ gives the matroid $M - m$. The set of vectors V/v_m, consisting of the projections $v_1', \ldots, v_{m-1}'$ of $v_1, \ldots, v_{m-1}$ onto the hyperplane E' orthogonal to v_m, gives the matroid M/m.

The generators v_I, $I \subseteq [m]$ of $\mathcal{S}_V$, fall into two categories: those for which $m \in I$, and those for which $m \notin I$. Let $\mathcal{S}_{V,1}$ and $\mathcal{S}_{V,2}$ be their spans, respectively.

$\mathcal{S}_{V,1}$ is generated by $v_I v_m$ for all $I \subseteq [m-1]$ such that $r([m]-(I \cup m)) = r$; that is, $r_{M-m}([m-1]-I) = r$. These are precisely the generators of $\mathcal{S}_{V-v_m}$ multiplied by v_m, so $\mathcal{S}_{V,1} = \mathcal{S}_{V-v_m} v_m$.

$\mathcal{S}_{V,2}$ is generated by v_I for all $I \subseteq [m-1]$ such that $r([m]-I) = r$; that is, $r_{M/m}([m-1]-I) = r-1$. Consider the projection map $\pi : E \to E'$ that sends $v_i \mapsto v'_i$ for $1 \leq i \leq m-1$. This map extends to a map $\pi : \mathrm{Sym}(E) \to \mathrm{Sym}(E')$, which sends generators of $\mathcal{S}_{V,2}$ to generators of $\mathcal{S}_{V/v_m}$; so $\pi(\mathcal{S}_{V,2}) = \mathcal{S}_{V/v_m}$. Let π' be the restriction of π to $\mathcal{S}_{V,2}$.

Now we make two observations. First, since $\mathcal{S}_{V,1}$ and $\mathcal{S}_{V,2}$ span $\mathcal{S}_V$, we have that $\dim \mathcal{S}_V = \dim \mathcal{S}_{V,1} + \dim \mathcal{S}_{V,2} - \dim(\mathcal{S}_{V,1} \cap \mathcal{S}_{V,2})$.

Second, $\pi(\mathcal{S}_{V,1} \cap \mathcal{S}_{V,2}) \subseteq \pi(\mathcal{S}_{V,1}) = 0$, since $\pi(v_m) = 0$. Therefore $\mathcal{S}_{V,1} \cap \mathcal{S}_{V,2} \subseteq \mathrm{Ker}\,\pi'$. It follows that $\dim(\mathcal{S}_{V,1} \cap \mathcal{S}_{V,2}) \leq \dim \mathrm{Ker}\,\pi' = \dim \mathcal{S}_{V,2} - \dim \mathcal{S}_{V/v_m}$.

These two observations give that $\dim \mathcal{S}_V \geq \dim \mathcal{S}_{V,1} + \dim \mathcal{S}_{V/v_m} = \mathrm{bases}(M - m) + \mathrm{bases}(M/m) = \mathrm{bases}(M)$, using the hypothesis of induction and the fact that $\mathrm{bases}(M)$ is a Tutte-Grothendieck invariant of M. But we already knew that $\dim \mathcal{S}_V \leq \mathrm{bases}(M)$, so the result follows. $\square$

4.3 The algebra $\mathcal{C}$

We will now consider four vector spaces and specify a basis for each one of them. Our collection of vectors V lies in the vector space $E = \Bbbk^n$. Let $F = \Bbbk^m$. Let $E^* = \mathrm{Hom}(E, \Bbbk)$ and $F^* = \mathrm{Hom}(F, \Bbbk)$ be the vector spaces dual to E and F, respectively.

Let $x^1, \ldots, x^n$ be a basis of E, and let $x_1, \ldots, x_n$ be the dual basis in E^*. Let $\phi^1, \ldots, \phi^m$ be a basis of F, and let $\phi_1, \ldots, \phi_m$ be the dual basis of F^*.

Consider the ideal $\mathfrak{I}$ of the symmetric algebra $\mathrm{Sym}(F^*)$ generated by the elements ϕ_i^2 for $1 \leq i \leq m$, and $\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}$ for $r_M([m] - \{i_1, \ldots, i_k\}) < n$. Let $\Phi_V = \mathrm{Sym}(F^*)/\mathfrak{I}$.

Consider the projection $p : F \to E$ such that $p(\phi^i) = v_i$ for $1 \leq i \leq m$. This projection induces a dual map $p^* : E^* \to F^*$, defined by $p^* e^*(f) = e^*(p\, f)$ for $e^* \in E^*$

and $f \in F$. This map induces an algebra homomorphism $p^* : \mathrm{Sym}(E^*) \to \mathrm{Sym}(F^*)$. Through composition with the canonical map $\mathrm{Sym}(F^*) \to \Phi_V$, we get a map $\tilde{p} : \mathrm{Sym}(E^*) \to \Phi_V$.

Definition 4.3.1 *Let $\mathcal{C}$ be the image of $\tilde{p}$ in Φ_V.*

Theorem 4.3.2 *We have* $\dim \mathcal{C} = \mathrm{bases}(M)$*. Furthermore, for $0 \le k \le m-n$, we have* $\dim \mathcal{C}^k = \mathrm{bases}(M, m-n-k)$*.*

In other words, the Hilbert-Poincaré series of $\mathcal{C}$ is $P(\mathcal{C}, t) = t^{m-n}T(1, \frac{1}{t})$.

Proof. We use a linear algebraic argument to prove that $\dim \mathcal{C}^k = \dim \mathcal{S}_V^k$, and then invoke Theorem 4.2.2.

The subspace $\mathcal{S}_V^k$ of $\mathrm{Sym}(E)$ is generated by the elements v_J, where J ranges over the set $\mathbb{J}_k = \{J \subseteq [m] \mid r_M([m] - J) = n, |J| = k\}$. A basis of $\mathrm{Sym}(E)$ is given by the set of elements $x^I = x^{i_1} \otimes \cdots \otimes x^{i_k}$, where $I = \{i_1 \le \ldots \le i_k\}$ ranges over the set $\mathbb{I}_k$ of k-multisets of $[m]$.

Now, for each $I \in \mathbb{I}_k, J \in \mathbb{J}_k$, let a_{IJ} be the coefficient of x^I in v_J; that is, let $v_J = \sum_{I \in \mathbb{I}_k} a_{IJ} x^I$. Let A_k be the matrix $(a_{IJ})_{I \in \mathbb{I}_k, J \in \mathbb{J}_k}$. Then we have

$$\dim \mathcal{S}_V^k = \mathrm{rank}\, A_k. \tag{4.3.1}$$

Now we do a similar analysis for $\mathcal{C}^k$. Let $p^* x_i = \theta_i \in F^*$. Since $x_1, \ldots, x_n$ generate E^*, $\mathcal{C}^k$ is generated by the elements $\theta_I = \theta_{i_1} \otimes \cdots \otimes \theta_{i_k}$ in Φ_V, where $I = \{i_1 \le \ldots \le i_k\}$ ranges over the set $\mathbb{I}_k$.

A basis of $\mathrm{Sym}(F^*)$ is given by the set of elements $\phi_J = \phi_{j_1} \otimes \cdots \otimes \phi_{j_k}$, where $J = \{j_1 \le \ldots \le j_k\}$ ranges over the set $\mathbb{I}_k$. From the relations that hold in Φ_V, a basis of Φ_V is given by the set of elements $\phi_J = \phi_{j_1} \otimes \cdots \otimes \phi_{j_k}$, where $J = \{j_1 < \ldots < j_k\}$ ranges precisely over the set $\mathbb{J}_k$. This is a consequence of the observation that, if $J \in \mathbb{I}_k - \mathbb{J}_k$, then $\phi_J = 0$ in Φ_V.

Now, for each $I \in \mathbb{I}_k, J \in \mathbb{J}_k$, let b_{IJ} be the coefficient of ϕ_J when we express θ_I in the basis $\{\phi_J \mid J \in \mathbb{J}_k\}$; that is, let $\theta_I = \sum_{J \in \mathbb{J}_k} b_{IJ} \phi_J$. Let B_k be the matrix

$(B_{IJ})_{I\in\mathbb{I}_k,\,J\in\mathbb{J}_k}$. Then we have

$$\dim\mathcal{C}^k = \operatorname{rank} B_k. \tag{4.3.2}$$

Let a_{ij} be the coefficient of x^i in v_j for $1 \le i \le n, 1 \le j \le m$, so $v_j = \sum_i a_{ij}x^i$ in E. Notice that $\theta_i(\phi^j) = p^*x_i(\phi^j) = x_i(p\,\phi^j) = x_i(v_j) = a_{ij}$. It follows that, in $\mathrm{Sym}(F^*)$, the coefficient of ϕ_j in θ_i is a_{ij}; *i.e.*, $\theta_i = \sum_j a_{ij}\phi_j$.

The reason that we are interested in the coefficients a_{ij} is that we can express the entries of A_k and B_k explicitly in terms of the them. We have that

$$\begin{aligned} v_J &= \left(\sum_{i_1=1}^{n} a_{i_1,j_1}x^{i_1}\right)\otimes\cdots\otimes\left(\sum_{i_k=1}^{n} a_{i_k,i_k}x^{j_k}\right) \\ &= \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1,j_1}\cdots a_{i_k,j_k}\ \left(x^{i_1}\otimes\cdots\otimes x^{i_k}\right) \\ &= \sum_{I=\{i_1\le\ldots\le i_k\}\in\mathbb{I}_k}\left(\sum_{\sigma\in\mathfrak{S}_I} a_{\sigma(i_1),j_1}\cdots a_{\sigma(i_k),j_k}\right)x^I, \end{aligned}$$

so

$$a_{IJ} = \sum_{\sigma\in\mathfrak{S}_I} a_{\sigma(i_1),j_1}\cdots a_{\sigma(i_k),j_k}. \tag{4.3.3}$$

Similarly,

$$\begin{aligned} \theta_I &= \left(\sum_{j_1=1}^{n} a_{i_1,j_1}\phi_{j_1}\right)\otimes\cdots\otimes\left(\sum_{j_k=1}^{n} a_{i_k,j_k}\phi_{j_k}\right) \\ &= \sum_{j_1,\ldots,j_k=1}^{n} a_{i_1,j_1}\cdots a_{i_k,j_k}\ \left(\phi_{j_1}\otimes\cdots\otimes\phi_{j_k}\right) \\ &= \sum_{J=\{j_1<\ldots<j_k\}\in\mathbb{J}_k}\left(\sum_{\sigma\in\mathfrak{S}_J} a_{i_1,\sigma(j_1)}\cdots a_{i_k,\sigma(j_k)}\right)\phi_J, \end{aligned}$$

so

$$b_{IJ} = \sum_{\sigma\in\mathfrak{S}_J} a_{i_1,\sigma(j_1)}\cdots a_{i_k,\sigma(j_k)}. \tag{4.3.4}$$

Let I consist of k_1 1's, k_2 2's, $\ldots$, k_n n's. The sum in (4.3.3) consists of $k!/k_I$

terms, where $k_I = k_1!k_2! \cdots k_n!$. It is clear that these terms are pairwise distinct as formal expressions in the a_{ij}'s.

The sum in (4.3.4) consists of $k!$ terms. Consider a term $a = a_{i_1,j_1} \cdots a_{j_k,i_k}$ in (4.3.3). For each value of i, there are k_i values of j for which a_{ij} appears in a. For each i, these k_i values can be permuted in any way without changing a; each one of these permutations is a term in (4.3.4). Therefore a appears exactly k_I times in (4.3.4).

We conclude that $b_{IJ} = k_I a_{IJ}$ for all $I \in \mathbb{I}_k$, $J \in \mathbb{J}_k$. Therefore the matrix B_k is obtained from the matrix A_k by multiplying row I by k_I for each $I \in \mathbb{I}_k$. Consequently, $\text{rank}(A_k) = \text{rank}(B_k)$. Theorem 4.3.2 then follows from (4.3.1) and (4.3.2). $\square$

Chapter 5

The Catalan matroid

5.1 Introduction

A *Dyck path* of length $2n$ is a path in the plane from $(0,0)$ to $(2n,0)$, with steps $(1,1)$ and $(1,-1)$, that never passes below the x-axis. It is a classical result (see for example [60, Corollary 6.2.3.(iv)]) that the number of Dyck paths of length $2n$ is equal to the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Each Dyck path P defines an *up-step set*, consisting of the integers i for which the i-th step of P is $(1,1)$. The starting point of this chapter is Theorem 5.2.1. It states that the collection of up-step sets of all Dyck paths of length $2n$ is the collection of bases of a matroid. Most of Chapter 5 is devoted to the study of this matroid, which we call the *Catalan matroid*, and denote $\mathbf{C}_n$.

Section 5.2 starts by proving Theorem 5.2.1. As we know, there are many equivalent ways of defining a matroid: in terms of its rank function, its independent sets, its flats, and its circuits, among others. The rest of Section 5.2 is devoted to describing some of these definitions for $\mathbf{C}_n$.

In Section 5.3, we compute the Tutte polynomial of the Catalan matroid. We find that it enumerates Dyck paths according to two simple statistics. Some nice enumerative results are derived as a consequence.

In Section 5.4, we generalize our construction of $\mathbf{C}_n$ to a wider class of matroids, which we call *shifted matroids*. They are precisely the matroids whose independence complex is a shifted simplicial complex. We then generalize our construction in a different direction to obtain, for any finite poset P and any order ideal I, a shifted family of sets. This family is not always the set of bases of a matroid.

Finally, in Section 5.5 we address the question of representability of the matroids we have constructed. We show that the Catalan matroid, and more generally any shifted matroid, is representable over $\mathbb{Q}$. In the opposite direction, we show that $\mathbf{C}_n$ is *not* representable over the finite field $\mathbb{F}_q$ if $q \leq n-2$.

Throughout Chapter 5, we will assume some familiarity with the basic concepts of matroid theory. For instance, Chapter 1 of [47] should be enough to understand most of the chapter. We also highly recommend Section 6.2 and Exercises 6.19–6.37 of [60] for an encyclopedic treatment of Catalan numbers and related topics.

5.2 The matroid

Let n be a fixed positive integer. Consider all paths in the plane which start at the origin and consist of $2n$ steps, where each step is either $(1,1)$ or $(1,-1)$. We will call such steps *up-steps* and *down-steps*, respectively. From now on, the word *path* will always to refer to a path of this form.

Such paths are in bijection with subsets of $[2n]$. To each path P, we can assign the set of integers i for which the i-th step of P is an up-step. We call this set the *up-step set* of P. Conversely, to each subset $A \subseteq [2n]$, we can assign the path whose i-th step is an up-step if and only if i is in A.

To simplify the notation later on, we will omit the brackets when we talk about subsets of $[2n]$. We will also use subsets of $[2n]$ and paths interchangeably. For example, for $n = 3$, the path 13 will be the path with up-steps at steps 1 and 3, and down-steps at steps $2, 4, 5$ and 6.

A useful statistic to keep track of will be the *height of path P at x*; *i.e.*, the height of the path after taking its first x steps. We shall denote it $\mathrm{ht}_P(x)$; it is equal to $2|P_{\leq x}| - x$, where $P_{\leq x}$ denotes the set of elements of P which are less than or equal to x. Also, let minht_P and maxht_P be the minimum and maximum heights that P

achieves, respectively.

Theorem 5.2.1 *Let $\mathcal{B}_n$ be the collection of up-step sets of all Dyck paths of length $2n$. Then $\mathcal{B}_n$ is the collection of bases of a matroid.*

Proof. We need to check the two axioms for the collection of bases of a matroid:

(B1) $\mathcal{B}_n$ is non-empty.

(B2) If A and B are members of $\mathcal{B}_n$ and $a \in A - B$, then there is an element $b \in B - A$ such that $(A - a) \cup b \in \mathcal{B}_n$.

The first axiom is satisfied trivially, so we only need to check the second one. Let A and B be members of $\mathcal{B}_n$, and let $a \in A - B$. First we will describe those k not in A for which $A - a \cup k \in \mathcal{B}_n$, and then we will show that the smallest element of $B - A$ is one of them.

For $k \notin A$, consider the path $A - a \cup k$, which is a very slight deformation of the path A. It still consists of n up-steps and n down-steps; to determine if it is a Dyck path, we just need to check whether it goes below the x-axis. There are two cases to consider.

The first case is that $k < a$. In this case, for $k \leq c < a$, we have that $\mathrm{ht}_{A - a \cup k}(c) = \mathrm{ht}_A(c) + 2$. For all other values of c, we have that $\mathrm{ht}_{A - a \cup k}(c) = \mathrm{ht}_A(c)$. Hence the path $A - a \cup k$ stays above the path A, so it is Dyck.

The second case is that $a < k$. Here, for $a \leq c < k$, we have that $\mathrm{ht}_{A - a \cup k}(c) = \mathrm{ht}_A(c) - 2$. For all other values of c, we have that $\mathrm{ht}_{A - a \cup k}(c) = \mathrm{ht}_A(c)$. Therefore, the path $A - a \cup k$ is Dyck if and only if $\mathrm{ht}_A(c) \geq 2$ for all $a \leq c < k$.

With that simple analysis, we can show that $A - a \cup b \in \mathcal{B}_n$, where b is the smallest element of $B - A$. If $b < a$, then we are done by the first case of our analysis. Otherwise, consider an arbitrary c with $a \leq c < b$. There are no elements of $B - A$ less than or equal to c; so up to the c-th step, every step which is an up-step in B is also an up-step in A. Furthermore, the a-th step is a down-step in B and an up-step in A. Therefore, $\mathrm{ht}_A(c) \geq \mathrm{ht}_B(c) + 2 \geq 2$. This concludes our proof. □

From Theorem 5.2.1, we have a unique matroid on the ground set $[2n]$ whose collection of bases is $\mathcal{B}_n$. We will call it the *Catalan matroid of rank* n (or simply the *Catalan matroid*), and denote it by $\mathbf{C}_n$. This chapter is mostly devoted to the study of this matroid.

Proposition 5.2.2 *The rank function of* $\mathbf{C}_n$ *is given by*

$$r(A) = n + \lfloor \operatorname{minht}_A / 2 \rfloor$$

for each $A \subseteq [2n]$.

Proof. Fix a subset $A \subseteq [2n]$, and let $\operatorname{minht}_A = -y$, where y is a non-negative integer. Also, let x be the smallest integer such that $\operatorname{ht}_A(x) = \operatorname{minht}_A$.

Recall that the rank of a subset A of $[2n]$ is equal to the largest possible size of an intersection $A \cap B$, where B is a basis of $\mathbf{C}_n$.

The path A is at height $-y$ after taking $|A_{\leq x}|$ up-steps and $x - |A_{\leq x}|$ down-steps, so $|A_{\leq x}| = (x-y)/2$. Also, for any basis B, we have that $|B_{>x}| \leq n - x/2$, since $\operatorname{ht}_B(x) \geq 0$. Hence

$$|A \cap B| = |(A \cap B)_{\leq x}| + |(A \cap B)_{>x}| \leq |A_{\leq x}| + |B_{>x}| \leq n - y/2.$$

We conclude that $r(A) \leq n + \lfloor \operatorname{minht}_A / 2 \rfloor$.

Now we need a basis B with $|A \cap B| = n + \lfloor \operatorname{minht}_A / 2 \rfloor$. We construct it as follows. First, add to A the smallest $a = \lceil y/2 \rceil$ numbers that it is missing, to obtain the set A'. Then $\operatorname{ht}_{A'}(x) = 2a - y \geq 0$; in fact, it is clear that the path A' never crosses the x-axis. Let $|A| = n + h$ for some integer h; then $\operatorname{ht}_A(2n) = 2h$ and $\operatorname{ht}_{A'}(2n) = 2h + 2a$. Now remove from A' the largest $h + a$ numbers that it contains, to obtain the set B. It is again easy to see that the path B never crosses the x-axis, and ends at $(2n, 0)$. So B is Dyck, and

$$|A \cap B| = |A \cap A'| - (h + a) = |A| - (h + a) = n - a$$

as desired. $\square$

Now that we know the rank function of $\mathbf{C}_n$, we describe several important classes of subsets of the matroid in Propositions 5.2.3 – 5.2.7. We will only provide a proof for Proposition 5.2.3; the remaining proofs are similar in flavor. The interested reader may want to complete the details to get better acquainted with the matroid $\mathbf{C}_n$.

Proposition 5.2.3 *The flats of $\mathbf{C}_n$ are the subsets $A \subseteq [2n]$ such that*

(i) minht_A *is odd, and*

(ii) if $\text{ht}_A(x) = \text{minht}_A$, *then* $\{x+1, \dots, 2n\} \subseteq A$.

Proof. Let A be a flat of $\mathbf{C}_n$, and let x be such that $\text{ht}_A(x) = \text{minht}_A$. If some integer y with $x+1 \leq y \leq n$ was not in A, then we would clearly have $\text{minht}_{A\cup y} = \text{minht}_A$ and thus $r(A \cup y) = r(A)$, contradicting the assumption that A is a flat. Therefore, any flat must satisfy condition *(ii)*.

Also, if we had a flat A with $\text{minht}_A = -2h$ achieved at $\text{ht}_A(x)$, then we would have $x \notin A$, and $\text{minht}_{A\cup x} = -2h+1$ would be achieved at $\text{ht}_{A\cup x}(x-1)$. We would then have $r(A \cup x) = r(A)$, again a contradiction. So any flat A must also satisfy condition *(i)*.

Conversely, assume that A satisfies conditions *(i)* and *(ii)*. Let $\text{minht}_A = -(2k+1)$, which can only be achieved once, say at $\text{ht}_A(x)$. Any y which is not in A must be less than or equal to x; and we have $\text{minht}_{A\cup y} = -(2k-1)$ if $y < x$, or $\text{minht}_{A\cup y} = -2k$ if $y = x$. In either case, $r(A \cup y) = r(A) + 1$. This completes the proof. $\square$

Proposition 5.2.4 *The independent sets of $\mathbf{C}_n$ are the subsets $A \subseteq [2n]$ such that* $\text{minht}_A = \text{ht}_A(2n)$.

Proposition 5.2.5 *The spanning sets of $\mathbf{C}_n$ are the subsets $A \subseteq [2n]$ such that* $\text{minht}_A = 0$.

Proposition 5.2.6 *The circuits of $\mathbf{C}_n$ are the subsets $A \subseteq [2n]$ of the form* $A = \{2k, 2k+b_1, \dots, 2k+b_{n-k}\}$, *for some positive integer $k \leq n$ and some Dyck path* $\{b_1, \dots, b_{n-k}\}$ *of length* $2(n-k)$.

Proposition 5.2.7 *The bonds of* $\mathbf{C}_n$ *are the subsets* $A \subseteq [2n]$ *such that*

(i) $\mathrm{maxht}_A = 1$, *and*

(ii) if $\mathrm{ht}_A(x) = 1$, *then* A *has no elements greater than* x.

We complete this section with an observation which is interesting in itself, and will also be important to us in section 5.3.

Proposition 5.2.8 *The Catalan matroid is self-dual.*[1]

Proof. Say $B = \{b_1, \ldots, b_n\}$ is a basis of $\mathbf{C}_n$, and let $[2n] - B = \{c_1, \ldots, c_n\}$ be the corresponding basis of the dual matroid $\mathbf{C}_n^*$. Then $\{2n+1-c_n, \ldots, 2n+1-c_1\}$ is a Dyck path; in fact, it is the path obtained by reflecting the Dyck path B across a vertical axis. So the bases of $\mathbf{C}_n^*$ are simply the up-step sets of all Dyck paths of length $2n$, under the relabeling $x \to 2n+1-x$. Thus $\mathbf{C}_n^* \cong \mathbf{C}_n$. $\square$

5.3 The Tutte polynomial

Given a matroid M over a ground set S, its Tutte polynomial is defined as:

$$T_M(q,t) = \sum_{A \subseteq S} (q-1)^{r(S)-r(A)}(t-1)^{|A|-r(A)}.$$

For our purposes, it is more convenient to define the Tutte polynomial in terms of the internal and external activity of the bases. We recall this definition now.

We first need to fix an arbitrary linear ordering of S.

For any basis B and any element $e \notin B$, the set $B \cup e$ contains a unique circuit. If e is the smallest element of that circuit with respect to our fixed linear order, then we say that e is *externally active* with respect to B. The number of externally active elements with respect to B is called the *external activity* of B; we shall denote it by $e(B)$.

[1]We follow Oxley [47] in calling a matroid M *self-dual* if $M \cong M^*$. It is worth mentioning, however, that some authors reserve the term 'self-dual' for matroids M such that $M = M^*$.

Dually, for any basis B and any element $i \in B$, the set $S - B \cup i$ contains a unique bond. If i is the smallest element of that bond, then we say that i is *internally active* with respect to B. The number of internally active elements with respect to B is called the *internal activity* of B; we shall denote it by $i(B)$.[2]

Proposition 5.3.1 *(Crapo, [20]) For any matroid M and any linear order of its ground set,*

$$T_M(q,t) = \sum_{B \text{ basis}} q^{i(B)} t^{e(B)}.$$

We will use Proposition 5.3.1 to study the Tutte polynomial of the Catalan matroid. The first thing to do is to fix a linear order of its ground set, $[2n]$. We will use the most natural choice: $1 < 2 < \cdots < 2n$. Now we compute the internal and external activity of each basis of $\mathbf{C}_n$.

Lemma 5.3.2 *The internal activity of a Dyck path B is equal to the number of up-steps that B takes before its first down-step.*

Proof. Let $i \in B$. The path $[2n] - B$ never goes above height 0; the path $[2n] - B \cup i$ goes up to height 2. Let j be the smallest integer such that $\mathrm{ht}_{[2n]-B\cup i}(j) = 1$. Clearly $j \geq i$.

Let D be the unique bond of $\mathbf{C}_n$ which can be obtained by deleting some elements of $[2n] - B \cup i$. We cannot delete any element less than or equal to j, or else the resulting path will not reach height 1. We must delete any element larger than j by Proposition 5.2.7. So $D = ([2n] - B)_{\leq j}$.

Therefore, i is the smallest element of D if and only if B contains all of $1, 2, \ldots, i-1$. This completes the proof. □

Lemma 5.3.3 *The external activity of a Dyck path B is equal to the number of positive integers x for which $\mathrm{ht}_B(x) = 0$.*

[2]The internally active elements with respect to a basis B of M are precisely the externally active elements with respect to the basis $S - B$ of the dual matroid M^*. That is why we say that internal activity and external activity are dual concepts.

Proof. Let $e \notin B$. The path $B \cup e$ ends at height 2; let $2k-1$ be the largest integer such that $\mathrm{ht}_{B\cup e}(2k-1) = 1$. Clearly $2k-1 < e$.

We start by showing that the unique circuit C of $\mathbf{C}_n$ contained in $B\cup e$ is $(B\cup e)_{\geq 2k}$.

Since $C \subseteq B\cup e$, we have that $\mathrm{ht}_C(2n) - \mathrm{ht}_C(2k-1) \leq \mathrm{ht}_{B\cup e}(2n) - \mathrm{ht}_{B\cup e}(2k-1) = 1$. Equality holds if and only if every up-step of $B \cup e$ after the $(2k-1)$-th is also an up-step of C; *i.e.*, when $(B \cup e)_{\geq 2k} = C_{\geq 2k}$.

But it is clear from Proposition 5.2.6 that $\mathrm{ht}_C(2n) - \mathrm{minht}_C = 1$, and that minht_C is only achieved at $\mathrm{ht}_C(\min C - 1)$. So the above inequality can only hold if $\min C = 2k$. Thus $C = C_{\geq 2k} = (B \cup e)_{\geq 2k}$ as desired.

Now we know that $\min C = 2k$, so e is externally active if and only if $e = 2k$. If $\mathrm{ht}_B(e) = 0$, this is clearly the case. On the other hand, if $\mathrm{ht}_B(e) \geq 1$, then $\mathrm{ht}_{B\cup e}(e-1) = \mathrm{ht}_B(e-1) \geq 2$, so this is not the case. This completes the proof. □

Theorem 5.3.4 *For a Dyck path P, let $a(P)$ denote the number of up-steps that P takes before its first down-step, and let $b(P)$ denote the number of positive integers x for which* $\mathrm{ht}_P(x) = 0$

Then the Tutte polynomial of the Catalan matroid $\mathbf{C}_n$ is equal to

$$T_{\mathbf{C}_n}(q,t) = \sum_{P \text{ Dyck}} q^{a(P)} t^{b(P)},$$

where the sum is over all Dyck paths of length $2n$.

Proof. This follows immediately from Proposition 5.3.1 and Lemmas 5.3.2 and 5.3.3.

Corollary 5.3.5 *The polynomial*

$$\sum_{P \text{ Dyck}} q^{a(P)} t^{b(P)},$$

is symmetric in q and t.

Proof. We know that, for any matroid M, the Tutte polynomial of its dual matroid is given by $T_{M^*}(q,t) = T_M(t,q)$. The result follows from Proposition 5.2.8 and Theorem 5.3.4. □

It is a known fact that the statistics $a(P)$ and $b(P)$ are equidistributed over the set of Dyck paths of length $2n$: the number of paths with $a(P)=k$ and the number of paths with $b(P)=k$ are both equal to $\frac{k}{2n-k}\binom{2n-k}{n}$. For the first equality, see for example [64]; for the second, see [38, eq. (7)].

Corollary 5.3.5 was also discovered independently by James Haglund [26]. It is not difficult to prove it directly; in fact, it will be an immediate consequence of our next theorem.

Theorem 5.3.6 *Let $C(x) = \frac{1}{2}\left(1-\sqrt{1-4x}\right) = C_0 + C_1x + C_2x^2 + \cdots$ be the generating function for the Catalan numbers. Then*

$$\sum_{n\geq 0} T_{\mathbf{C}_n}(q,t)x^n = \frac{1+(qt-q-t)xC(x)}{1-qtx+(qt-q-t)xC(x)}.$$

Proof. A Dyck path P of length $2n \geq 2$ can be decomposed uniquely in the standard way: it starts with an up-step, then it follows a Dyck path P_1 of length $2r$, then it takes a down-step, and it ends with a Dyck path P_2 of length $2s$, for some non-negative integers r,s with $r+s=n-1$. More precisely, and necessarily more confusingly,

$$P = \{1, 1+p_1, 1+p_2, \ldots, 1+p_r, 2r+2+q_1, 2r+2+q_2, \ldots, 2r+2+q_s\}$$

for some Dyck paths $\{p_1,\ldots,p_r\}$ and $\{q_1,\ldots,q_s\}$ with $r+s=n-1$.

It is clear that in this decomposition we have $a(P) = a(P_1)+1$ and $b(P) = b(P_2)+1$. Therefore

$$\begin{aligned} T_{\mathbf{C}_n}(q,t) &= \sum_{r+s=n-1}\ \sum_{P_1\,\in\,\mathcal{B}_r}\ \sum_{P_2\,\in\,\mathcal{B}_s} q^{a(P_1)+1}\,t^{b(P_2)+1} \\ &= qt\sum_{r+s=n-1} T_{\mathbf{C}_r}(q,1)T_{\mathbf{C}_s}(1,t) \end{aligned}$$

for $n \geq 1$; so if we write $\mathbf{T}(q,t,x) = \sum_{n\geq 0} T_{\mathbf{C}_n}(q,t)x^n$, we have

$$\mathbf{T}(q,t,x) = 1 + qtx\mathbf{T}(q,1,x)\mathbf{T}(1,t,x). \tag{5.3.1}$$

Now observe that $\mathbf{T}(1,1,x) = C(x)$. Setting $q = 1$ in (5.3.1) gives a formula for $\mathbf{T}(1,t,x)$, and setting $t = 1$ gives a formula for $\mathbf{T}(q,1,x)$. Substituting these two formulas back into (5.3.1), we get the desired result. □

5.4 Shifted matroids

We now generalize our construction of $\mathbf{C}_n$ to a larger family of matroids, which we call *shifted matroids.* There is one shifted matroid for each non-empty set $S = \{s_1 < \cdots < s_n\}$ of positive integers, which we shall denote $\mathbf{SM}(s_1, \ldots, s_n)$

Theorem 5.4.1 *Let $S = \{s_1 < \cdots < s_n\}$ be a set of positive integers, and let $\mathcal{B}_S$ be the collection of sets $\{a_1 < \cdots < a_n\}$ such that $a_1 \leq s_1, \ldots, a_n \leq s_n$. Then $\mathcal{B}_S$ is the collection of bases of a matroid $\mathbf{SM}(s_1, \ldots, s_n)$.*

Proof. Once again, as in the proof of Theorem 5.2.1, axiom **(B1)** is trivial, since $S \in B_S$. We need to check axiom **(B2)**. Let $A = \{a_1 < \cdots < a_n\}$ and $B = \{b_1 < \cdots < b_n\}$ be in B_S, and let $a_x \in A - B$. We claim that, if b_y is the smallest element in $B - A$, then $A - a_x \cup b_y \in \mathcal{B}_S$.

Let i be the integer such that $a_i < b_y < a_{i+1}$. (If $b_y < a_1$ then the claim is trivially true, since we are replacing a_x in A with a number smaller than it. If $b_y > a_n$ then set $i = n$.) We may assume that $i \geq x$; if that was not the case, then we would have $b_y < a_{i+1} \leq a_x$, and the claim would be trivial. We then have

$$A - a_x \cup b_y = \{a_1 < \cdots < a_{x-1} < a_{x+1} < \cdots < a_i < b_y < a_{i+1} < \cdots < a_n\}$$

and we have n inequalities to check.

The first $x - 1$ and the last $n - i$ of these do not require any extra work: we already know that $a_k \leq s_k$ for $1 \leq k \leq x - 1$ and for $i + 1 \leq k \leq n$.

For each value of k with $x \leq k \leq i-1$, we need to check that $a_{k+1} \leq s_k$. If $k \geq y$, we have $a_{k+1} \leq a_i < b_y \leq b_k \leq s_k$. Otherwise, if $k < y$, proceed as follows. Since b_y is the smallest element of B which is not in A, and a_x is not in B, the numbers $b_1, \ldots, b_k$ must all be somewhere in the list $a_1, \ldots, a_{x-1}, a_{x+1}, \ldots a_i$. Therefore the k-th smallest number of this list, a_{k+1}, must be less than or equal to b_k, which is less than or equal to s_k.

Finally, we need to check that $b_y \leq s_i$. Since the numbers $b_1, \ldots, b_{y-1}$ all appear in the list $a_1, \ldots, a_{x-1}, a_{x+1}, \ldots, a_k, \ldots, a_i$, we have $y-1 \leq i-1$. Therefore $b_y \leq s_y \leq s_i$. □

The Catalan matroid is a member of the shifted matroid family. A path $\{a_1 < \cdots < a_n\}$ is Dyck if and only if, for each i with $1 \leq i \leq n$, the i-th up-step comes before the i-th down-step; that is, if and only if $a_i \leq 2i-1$. Therefore, the Catalan matroid $\mathbf{C}_n$ is exactly the shifted matroid $\mathbf{SM}(1, 3, 5, \ldots, 2n-1)$, with an additional loop $2n$.

Shifted matroids have been discovered several times in the past. Welsh [69] used them to prove a lower bound for the number of non-isomorphic matroids on $[n]$. Oxley, Prendergast and Row [46] gave different characterizations of them. Bonin, de Mier and Noy [12], [13] are currently studying a slightly wider class of matroids with very nice structural and enumerative properties.

Now we present a new characterization of shifted matroids. Recall that an *abstract simplicial complex* Δ on $[n]$ is a family of subsets of $[n]$ (called *faces*) such that if $G \in \Delta$ and $F \subseteq G$, then $F \in \Delta$. A simplicial complex Δ is *shifted* if, for any face $F \in \Delta$ and any pair of elements $i < j$ such that $i \notin F$ and $j \in F$, the subset $F - j \cup i$ is also a face of Δ.

The family of independent sets of a matroid M is always a simplicial complex, called the *independence complex* or *matroid complex* of M. For shifted matroids, we have the following simple observation.

Proposition 5.4.2 *For any positive integers $s_1 < \cdots < s_n$, the independence complex of the shifted matroid $\mathbf{SM}(s_1, \ldots, s_n)$ is a shifted complex.*

Proof. If $F \subseteq [s_n]$ is independent, it is contained in some basis B. Now assume that we have two elements $i < j$ such that $i \notin F$ and $j \in F$, and let $G = F - j \cup i$. We need to show that G is independent. If the basis B contains i, then it contains G. Otherwise, $B - j \cup i$ is also a basis: for any $1 \leq k \leq n$, its k-th smallest is less than or equal to the k-th smallest element of B, which is less than or equal to s_k. This basis contains G. In both cases, we conclude that G is independent. □

In [35], Klivans characterizes *shifted matroid complexes*: shifted complexes which are independence complexes of matroids. Her results and ours were discovered almost simultaneously. When we sat down to discuss them, we realized that her shifted matroid complexes were precisely the independence complexes of the matroids $\mathbf{SM}(s_1, \ldots, s_n)$. This is why these matroids were baptized "shifted matroids".

Proposition 5.4.3 *(Klivans, [35]) If the independence complex of a loop-less matroid M is a shifted complex, then $M \cong \mathbf{SM}(s_1, \ldots, s_n)$ for some positive integers $s_1 < \cdots < s_n$.*

Corollary 5.4.4 *The independence complex of a matroid M is a shifted complex if and only if M is a shifted matroid.*

One of the main reasons of interest in studying shifted complexes is the simplicity of their topology. To any simplicial complex Γ, one can associate a shifted simplicial complex $\Delta(\Gamma)$. This shifted complex preserves many combinatorial and topological properties of Γ, but is much easier to study [32]. In particular, any shifted complex is homotopically equivalent to a wedge of spheres. Therefore, its homology groups have no torsion and its cohomology ring is trivial.

We now use these facts to discuss the topology of Catalan matroid complexes and, more generally, shifted matroid complexes. The Catalan matroid complex is contractible, since every basis contains 1. So is the complex of any shifted matroid $\mathbf{SM}(1, s_2, \ldots, s_n)$. Instead, consider the *reduced Catalan matroid complex*: the independence complex of the Catalan matroid with the coloop 1 deleted.

Proposition 5.4.5 *The reduced Catalan matroid complex is homotopically equivalent to a wedge of C_{n-1} $(n-2)$-dimensional spheres.*

Let $s_1 < \ldots < s_n$ be integers with $s_1 > 1$. The independence complex of the shifted matroid $\mathbf{SM}(s_1, \ldots, s_n)$ is homotopically equivalent to a wedge of $(n-1)$-dimensional spheres. The number of spheres is equal to the number of bases of the shifted matroid $\mathbf{SM}(s_1 - 1, \ldots, s_n - 1)$.

Proof. We use Lemma 3.1 of [32]. Any shifted complex is homotopically equivalent to a wedge of spheres, possibly of different dimensions. In our case, since matroid complexes are pure, the spheres must have the same dimension.

For the reduced Catalan matroid complex, the spheres are $(n-2)$-dimensional. The number of them is equal to the number of maximal faces of the complex which do not contain 2; that is, the number of Dyck paths of length $2n$ whose second step is a down-step. There are C_{n-1} such paths.

For the independence complex of $\mathbf{SM}(s_1, \ldots, s_n)$, the spheres are $(n-1)$-dimensional. The number of them is equal to the number of maximal faces of the complex which do not contain 1. Subtracting 1 from the labels of these faces puts them in bijective correspondence with the bases of $\mathbf{SM}(s_1 - 1, \ldots, s_n - 1)$. □

Theorem 5.4.1 and Propositions 5.4.2 and 5.4.3 have a nice application to Young tableaux. Recall that a *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n is a weakly decreasing sequence of positive integers which add up to n. We associate to it a *Young diagram*: a left-justified array of unit squares, which has λ_i squares on the i-th row from top to bottom.[3] A *standard Young tableaux* is a placement of the integers $1, \ldots, n$ in the squares of the Young diagram, in such a way that the numbers are increasing from left to right and from top to bottom.

These definitions will be sufficient for our purposes. For a much deeper treatment of the theory of Young tableaux, we refer the reader to [24].

Corollary 5.4.6 *Let λ be a partition. Define the* first row set *of a standard Young tableau T of shape λ to be the set of entries which appear in the first row of T. Then the collection of first row sets of all standard Young tableaux of shape λ is the collection of bases of a shifted matroid.*

[3]This is the English way of drawing Young diagrams; francophones draw them with λ_i squares on the i-th row from bottom to top.

Proof. Let $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$ be the conjugate partition of λ, so λ'_i is the number of squares on the i-th column of the Young diagram of λ. Let $s_i = 1 + \lambda'_1 + \cdots + \lambda'_{i-1}$ for $1 \leq i \leq n$.

Let $\{b_1 < \cdots < b_n\}$ be the first row set of a standard Young tableau T of shape λ. The first entry on the i-th column of T is b_i; it is smaller than every entry to its southeast. There are only $\lambda'_1 + \cdots + \lambda'_{i-1}$ cells which are not to its southeast, so $b_i \leq s_i$.

Conversely, if $B = \{b_1 < \cdots < b_n\}$ is such that $b_i \leq s_i$ for $1 \leq i \leq n$, then we can construct a standard Young tableau with first row set B. To do it, we first put the elements of B in order on the first row of λ. Then we put the remaining numbers from 1 to $|\lambda|$ on the remaining cells going in order down the columns, starting with the leftmost column. The inequalities $b_i \leq s_i$ guarantee that this process does indeed give a Young tableau T.

It follows that the collection in question is simply the collection of bases of the matroid $\mathbf{SM}(s_1, \ldots, s_n)$. □

We might try to generalize Corollary 5.4.6, replacing the first row of λ by any partition $\mu \subseteq \lambda$. Define the *μ-set* of a standard Young tableau T of shape λ to be the set of entries which appear in the sub-shape μ in T.

It is not too difficult to see that we do not always get the collection of bases of a matroid with this construction. However, we can still say something interesting.

Proposition 5.4.7 *Let $\mu \subseteq \lambda$ be partitions. Then the collection $\mathcal{B}_{\lambda\mu}$ of μ-sets of all standard Young tableau of shape λ is a shifted family.*

In fact, we prove something more general. Let P be a poset of n elements. Recall that a subset I of P is an *order ideal* of P if, for any pair of elements $x, y \in P$ with $x <_P y$ and $y \in I$, we also have $x \in I$. Also recall that a *linear extension* of P is a bijection $f : P \to [n]$ such that $i <_P j$ implies that $f(i) < f(j)$. It is convenient to think of a linear extension of P as a labeling of its elements satisfying the given condition. For more information on posets, we refer the reader to [56, Chapter 3].

Define the *I-set* of a linear extension f of P to be the set $\{f(i) : i \in I\}$.

Proposition 5.4.8 *Let P be a poset and let I be an order ideal of P. Then the collection $\mathcal{B}_{P,I}$ of I-sets of all linear extensions of P is a shifted family.*

Proof of Proposition 5.4.8. We need to check that if we have a set $B \in \mathcal{B}_{P,I}$ and a pair of numbers $a < b$ such that $a \notin B$ and $b \in B$, then $B - b \cup a \in \mathcal{B}_{P,I}$. It is enough to show this for $a = b - 1$; the general case will then follow by induction on $b - a$.

So let f be a linear extension of P with I-set B, and let $b \in B$ be such that $b - 1 \notin B$. Let $b = f(i)$ and $b - 1 = f(p)$ where $i \in I$ and $p \in P - I$. Let $g : P \to [n]$ be defined by switching the values of f at i and p; *i.e.*,

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \{i, p\} \\ b - 1 & \text{if } x = i \\ b & \text{if } x = p \end{cases} \tag{5.4.1}$$

We claim that g is also a linear extension for P. An important observation is that i and p are incomparable in P. If we had $i < p$, then we would have $b = f(i) < f(p) = b - 1$. If we had $i > p$, then $i \in I$ would imply $p \in I$.

To check that f is a linear extension, we need to check that $f(i) = b$ satisfies several inequalities: it must be greater than all the values that f takes on $P_{<i}$, and less than all the values that f takes on $P_{>i}$. But b is never compared to $b - 1$ here, since p and i are incomparable. Therefore, $b - 1$ also satisfies all those inequalities that b needs to satisfy.

Similarly, $b - 1$ must be greater than all the values that f takes on $P_{<p}$ and less than all the values that f takes on $P_{>p}$. The number b also satisfies these inequalities.

So we can switch the values of $f(i)$ and $f(p)$, and the resulting function g defined by (5.4.1) will also be a linear extension of P. Also, the I-set of g is $B - b \cup (b - 1)$. This concludes the proof. $\square$

Proof of Proposition 5.4.7. The cells of λ can be given a natural partial order P_λ: cell i is less than cell j in P_λ if and only if cell i is northwest of cell j in λ. The cells of μ define an order ideal I_μ of P_λ, and $\mathcal{B}_{\lambda\mu} = \mathcal{B}_{P_\lambda, I_\mu}$. Now use Proposition 5.4.8. $\square$

In view of Corollary 5.4.6 and Proposition 5.4.8, a natural problem, suggested by Richard Stanley, is the following.

Problem 5.4.9 *Characterize the pairs (P, I) of a finite poset P and an order ideal I for which $B_{P,I}$ is the set of bases of a matroid.*

5.5 Representability

A natural question to ask is whether the Catalan matroid can be represented as the vector matroid of a collection of vectors. We answer that question in this section.

Given a collection of real numbers $x_1, \ldots, x_k$, let $x_S = \prod_{i \in S} x_i$ for each subset $S \subseteq [k]$. Form all the 2^{2^k} possible sums of some of the x_S's. If these sums are all distinct, we will say that the initial collection of numbers is *generic*. Most collections of real numbers are generic. A specific example is a set of algebraically independent real numbers. Another example is any sequence of positive integers which increases quickly enough; for instance, one that satisfies $x_1 > 1$ and $x_i > (1+x_1)(1+x_2)\cdots(1+x_{i-1})$ for $1 < i \leq k$.

Theorem 5.5.1 *Let $v_1, \ldots, v_{2n}$ be the columns of a matrix*

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & \ldots & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & a_{n6} & \ldots & a_{n,2n-1} & 0 \end{pmatrix}$$

where the a_{ij}'s with $1 \leq i \leq n$ and $1 \leq j \leq 2i-1$ are generic integers. Then the vector matroid of $\{v_1, \ldots, v_n\}$ is isomorphic to the Catalan matroid $\mathbf{C}_n$.

Proof. Let M be the vector matroid of V. Let $1 \leq b_1 < \cdots < b_n \leq 2n$. The set $B = \{v_{b_1}, \ldots, v_{b_n}\}$ is a basis for M if and only if it is independent; that is, if and only if the determinant of the matrix A_B with columns $v_{b_1}, \ldots, v_{b_n}$ is non-zero.

This determinant is a sum of $n!$ terms, with plus or minus signs attached to them. Since the a_{ij}'s are generic, this sum can only be zero if all the terms are 0. So B is a basis as long as at least one of the $n!$ terms in this determinant is non-zero.

The question is now whether it is possible to place n non-attacking rooks on the non-zero entries of A_B; that is, to choose n non-zero entries with no two on the same row or column. The marriage theorem [65, Theorem 5.1] would be the standard tool to attack this kind of question. However, A_B is such that any entry below or to the left of a non-zero entry is also non-zero. This fact will make our argument shorter and self-contained.

If $b_i \leq 2i-1$ for all integers i with $1 \leq i \leq n$, then the (i,i) entry of A_B is $a_{i,b_i} \neq 0$. Therefore we can place n non-attacking rooks on non-zero entries of A_B by putting them on the main diagonal.

Conversely, suppose that we have a placement of n non-attacking rooks on non-zero entries of A_B. Let i be any integer between 1 and n. Then the rooks on the first i rows must be on i different columns. From the shape that the non-zero entries of A_B form, we conclude that the i-th row must contain i different non-zero entries. Thus the (i,i) entry of A_B, which is precisely a_{i,b_i}, must be non-zero. Therefore $b_i \leq 2i-1$.
□

The above proof essentially shows that $\mathbf{C}_n$ is a transversal matroid, with presentation $([1],[3],[5],\ldots,[2n-1])$. It generalizes immediately to any shifted matroid $\mathbf{SM}(s_1,\ldots,s_n)$ as follows.

Theorem 5.5.2 *Let $s_1 < \cdots < s_n$ be positive integers. Let $v_1,\ldots,v_{s_n}$ be the columns of a matrix $A = (a_{ij})_{1\leq i\leq n,\, 1\leq j\leq s_n}$, where the a_{ij}'s with $1 \leq i \leq n$ and $1 \leq j \leq s_i$ are generic, and the remaining a_{ij}'s are equal to 0. Then the vector matroid of $\{v_1,\ldots,v_{s_n}\}$ is isomorphic to the shifted matroid $\mathbf{SM}(s_1,\ldots,s_n)$.*

Theorem 5.5.1 shows that the Catalan matroid is representable over $\mathbb{Q}$, or even over a sufficiently large finite field. In the other direction, we now show a negative result about representing $\mathbf{C}_n$ over finite fields.

Proposition 5.5.3 *The Catalan matroid* $\mathbf{C}_n$ *is* not *representable over the finite field* $\mathbb{F}_q$ *if* $q \leq n-2$.

Proof. It is known ([47], Proposition 6.5.2) and easy to show that the uniform matroid $U_{2,k}$ is $\mathbb{F}_q$-representable if and only if $q \geq k-1$. A matroid containing it as a minor is not representable over $\mathbb{F}_q$ for $q \leq k-2$. This suggests that we should find the largest k for which $U_{2,k}$ is a minor of $\mathbf{C}_n$.

We can use the Scum theorem (Higgs, [47], Proposition 3.3.7), which essentially says that, if a matroid has a certain minor, then it must have that minor hanging from the top of its lattice of flats. Our question is then equivalent to finding the largest k for which there exists a rank - $(n-2)$ flat which is contained in k rank - $(n-1)$ flats.

Lemma 5.5.4 *Let A be a rank - $(n-2)$ flat, and let x be the smallest integer such that* $\mathrm{ht}_A(x) = -1$. *Then there are exactly* $\frac{x+3}{2}$ *rank - $(n-1)$ flats containing A.*

Proof of Lemma 5.5.4. We know from Propositions 5.2.2 and 5.2.3 that $\mathrm{minht}_A = -3$ and that, once the path A reaches height -3, say at $\mathrm{ht}_A(y)$, it only takes up-steps. We want to add elements to A to obtain a path which reaches a minimum height -1, and only takes up-steps after that.

Say that we add one element a to A. This new up-step at a comes before the y-th, so $\mathrm{ht}_{A\cup a}(y) = -1$. If we don't want to add any more elements to A, we have to make sure that $A \cup a$ only reaches height -1 at y. For this to be true, we need the new up-step a to occur on or before the x-th step. In A, there are $\frac{x+1}{2}$ down-steps up to the x-th to choose from. Each one of these gives a rank - $(n-1)$ flat containing A.

On the other hand, if we are to add more elements to A to obtain a rank-$(r-1)$ flat B, they will all be less than y so we will have $\mathrm{ht}_B(y) > 0$. The minimum height in B must then be achieved at some z for which $\mathrm{ht}_A(z) = -1$. In fact, for this z to be unique, it must be the leftmost one, *i.e.*, it must be x. So the only possibility is that $B = A_{\leq x} \cup \{x+1, \ldots, 2n\}$, which is indeed a rank - $(n-1)$ flat. This concludes the proof of Lemma 5.5.4. □

Having shown Lemma 5.5.4, the rest is easy. The rank - $(n-2)$ flat which is contained in the largest number of rank - $(n-1)$ flats, is the latest one to arrive to height -1. This flat is clearly $\{1, 2, \ldots, n-3, n-2, 2n\}$, which arrives to height -1 after $2n-3$ steps. It is contained in exactly n rank-$(n-1)$ flats.

Therefore $\mathbf{C}_n$ contains $U_{2,n}$ as a minor, and thus it is not representable over a field $\mathbb{F}_q$ with $q \leq n-2$. $\square$

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