The equivariant Ehrhart theory of the permutahedron

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Abstract

Equivariant Ehrhart theory enumerates the lattice points in a polytope with respect to a group action. Answering a question of Stapledon, we describe the equivariant Ehrhart theory of the permutahedron, and we prove his Effectiveness Conjecture in this special case.

1 Introduction

Ehrhart theory measures a polytope $P$ by counting the lattice points in its dilations $tP$ for $t \in \mathbb{N}$. Stapledon [11] introduced equivariant Ehrhart theory as a refinement of Ehrhart theory that takes into account the symmetries of the polytope $P$. He asked for a description of the equivariant Ehrhart theory of the permutahedron under its group of symmetries, the symmetric group. This study was initiated in [2], which computed the equivariant volumes of the permutahedron. In this paper we complete the answer to Stapledon’s question, computing the equivariant Ehrhart polynomials of the permutahedron, and verifying several conjectures in this special case.

1.1 The Ehrhart quasipolynomials of the fixed polytopes of the permutahedron

We consider the action of the symmetric group $S_n$ on the $(n-1)$-dimensional permutahedron $\Pi_n$. For each permutation $\sigma \in S_n$, we define the fixed polytope $\Pi_n^\sigma \subseteq \Pi_n$ to be the subset of the permutahedron $\Pi_n$ fixed by $\sigma$. Our first main result is a combinatorial formula for its lattice-point enumerator $L_{\Pi_n^\sigma}(t) := |t\Pi_n^\sigma \cap \mathbb{Z}^n|:

Theorem 1.1. Let $\sigma$ be a permutation of $[n]$ and let $\lambda = (\ell_1, \ldots, \ell_m)$ be the partition of $[n]$ given by the lengths of the cycles of $\sigma$. Say a set partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$ is $\lambda$-compatible if for each block $B_i$, either $\ell_j$ is odd for some $j \in B_i$, or the minimum 2-valuation among $\{\ell_j : j \in B_i\}$ is attained at least twice. Also write

$$v_{\pi} = \prod_{i=1}^{k} \left( \frac{\gcd_{j \in B_i} \ell_j \cdot \left( \sum_{j \in B_i} \ell_j \right)^{|B_i|-2}}{\prod_{j \in B_i} \ell_j} \right).$$

Then the Ehrhart quasipolynomial of the fixed polytope of the permutahedron $\Pi_n$ fixed by $\sigma$ is

$$L_{\Pi_n^\sigma}(t) = \begin{cases} \sum_{\pi \models [m]} v_{\pi} \cdot t^{m-|\pi|} & \text{if } t \text{ is even} \\ \sum_{\pi \models [m]} v_{\pi} \cdot t^{m-|\pi|} & \text{if } t \text{ is odd} \end{cases}$$

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The authors were supported by NSF Award DMS-1600609 and DMS-1855610 and a Simons Fellowship (FA), the National Physical Science Consortium (MS), and NSF Graduate Research Fellowship DGE-1247392 (ARVM).
### 1.2 Equivariant Ehrhart theory

Theorem 1.1 fits into the framework of equivariant Ehrhart theory, as we now explain.

Let $G$ be a finite group acting on $\mathbb{Z}^n$ and $P \subseteq \mathbb{R}^n$ be a $d$-dimensional lattice polytope that is invariant under the action of $G$. Let $M$ be the sublattice of $\mathbb{Z}^n$ obtained by translating the affine span of $P$ shifted to the origin, and consider the induced representation $\rho : G \to GL(M)$. We then obtain a family of permutation representations by looking at how $\rho$ permutes the lattice points inside the dilations of $P$. Let $\chi_{\ast P} : G \to \mathbb{C}$ denote the permutation character associated to the action of $G$ on the lattice points in the $t^{\text{th}}$ dilate of $P$. We have 

$$\chi_{\ast P}(g) = L_{P^g}(t)$$

where $P^g$ is the polytope of points in $P$ fixed by $g$ and $L_{P^g}(t)$ is its lattice point enumerator.

The permutation characters $\chi_{\ast P}$ live in the ring $R(G)$ of virtual characters of $G$, which are the integer combinations of the irreducible characters of $G$. The positive integer combinations are called effective; they are the characters of representations of $G$.

Stapledon encoded the characters $\chi_{\ast P}$ in a power series $H^\ast[z] \in R(G)[[z]]$ given by

$$\sum_{t \geq 0} \chi_{\ast P}(g)z^t = \frac{H^\ast[z](g)}{(1-z)\det(I-g\cdot z)}.$$  

(2)

We say that $H^\ast[z] := \sum_{t \geq 0} H^\ast_t z^t$ is effective if each virtual character $H^\ast_t$ is a character. Stapledon denoted this series $\varphi[t]$, but we denote it $H^\ast[z]$ and call it the equivariant $H^\ast$-series because for the identity element, $H^\ast[z](e) = h^\ast[z]$ is the well-studied $h^\ast$-polynomial of $P$.

The main open problem in equivariant Ehrhart theory is to characterize when $H^\ast[z]$ is effective, and Stapledon offered the following conjecture.

**Conjecture 1.2** ([11, Effectiveness Conjecture 12.1]). Let $P$ be a lattice polytope fixed by the action of a group $G$. The following conditions are equivalent.

(i) The toric variety of $P$ admits a $G$-invariant non-degenerate hypersurface.

(ii) The equivariant $H^\ast$-series of $P$ is effective.

(iii) The equivariant $H^\ast$-series of $P$ is a polynomial.

Our second main result is the following.

**Theorem 1.3.** Stapledon’s Effectiveness Conjecture holds for the permutahedron under the action of the symmetric group.

Finally, in Propositions ??, ??, and ?? we verify the three remaining conjectures of Stapledon in this case. There is some surprisingly subtle number theory at play in most of our results, as the following examples illustrate.

### 1.3 Examples

**Example 1.4.** Let us illustrate these results for the permutahedron $\Pi_4$ and the permutation $\sigma = (12)(3)(4)$ which has cycle type $\lambda = (2,1,1)$, illustrated in Figure 1. The fixed polytope $\Pi_4^{(12)}$ is a half-integral hexagon, and one may verify manually that

$$L_{\Pi_4^{(12)}}(t) = \begin{cases} 4t^2 + 3t + 1 & \text{if } t \text{ is even} \\ 4t^2 + 2t & \text{if } t \text{ is odd}, \end{cases} \quad H^\ast[z](12) = 1 + 4z + 11z^2 - 2z^3 + \frac{4z^4}{1+z}.$$ 

Since the $H^\ast$-series of $\Pi_4$ is not polynomial when evaluated at (12), Stapledon’s Conjecture 1.2 predicts that it is also not effective, and that the permutahedral variety $X_{\Pi_4}$ does not admit an $S_4$-invariant non-degenerate hypersurface. We verify this in Section 5.

The equivariant Ehrhart quasipolynomials and $H^\ast$-series of $\Pi_3$ and $\Pi_4$ are shown in Tables 2 and 3.
Example 1.5. Further subtleties already arise in the simple case when $\Pi_\sigma^n$ is a segment; this happens when $\sigma$ has only two cycles of lengths $\ell_1$ and $\ell_2$. For even $t$, we simply have

$$L_{\Pi_\sigma^n}(t) = \gcd(\ell_1, \ell_2)t + 1.$$  

However, for odd $t$ we have

$$L_{\Pi_\sigma^n}(t) = \begin{cases} 
\gcd(\ell_1, \ell_2)t + 1 & \text{if } \ell_1 \text{ and } \ell_2 \text{ are both odd}, \\
\gcd(\ell_1, \ell_2)t & \text{if } \ell_1 \text{ and } \ell_2 \text{ have different parity}, \\
\gcd(\ell_1, \ell_2)t & \text{if } \ell_1 \text{ and } \ell_2 \text{ are both even and they have the same 2-valuation}, \\
0 & \text{if } \ell_1 \text{ and } \ell_2 \text{ are both even and they have different 2-valuations}.
\end{cases}$$

We invite the reader to verify that this formula follows from Theorem 1.1.

1.4 Organization

In Section 2 we introduce some background on Ehrhart theory and zonotopes. In Section 3 we compute the Ehrhart polynomial of the fixed polytope $\Pi_\sigma^n$ when it is a lattice polytope, and in Section 4 we compute its Ehrhart quasipolynomial in general, proving Theorem 1.1. In Section 5 we compute the equivariant $H^*$-series $H^*[z]$ for permutahedra and we verify Stapledon’s four conjectures on equivariant Ehrhart theory in this special case; most importantly, his Effectiveness Conjecture (Theorem 1.3).

2 Preliminaries

2.1 Ehrhart quasipolynomials

Let $P$ be a convex polytope in $\mathbb{R}^n$. We say that $P$ is a rational polytope if all of its vertices are in $\mathbb{Q}^n$. The lattice point enumerator of $P$ is the function $L_P : \mathbb{N} \to \mathbb{N}$ given by

$$L_P(t) := |tP \cap \mathbb{Z}^n|;$$

that is, $L_P(t)$ is the number of integer points in the $t$th dilate of $P$. A function $f : \mathbb{Z} \to \mathbb{R}$ is quasipolynomial if there exists a period $d$ and polynomials $f_0, f_1, \ldots, f_{d-1}$ such that $f(n) = f_i(n)$ whenever $n \equiv i \pmod{d}$.

**Theorem 2.1** (Ehrhart’s Theorem [4]). If $P$ is a rational polytope, then $L_P(t) := |tP \cap \mathbb{Z}^n|$ is a quasipolynomial in $t$. Its degree is $\dim P$ and its period divides the least common multiple of the denominators of the coordinates of the vertices of $P$. 

Figure 1: The fixed polytope $\Pi_4^{(12)}$ is a half-integral hexagon containing 6 lattice points.
2.2 Zonotopes

Let $V$ be a finite set of vectors in $\mathbb{R}^n$. The zonotope generated by $V$, denoted $Z(V)$, is defined to be the Minkowski sum of the line segments connecting the origin to $v$ for each $v \in V$. We will also adopt the same notation to refer to any translation of $Z(V)$, that is, the Minkowski sum of any collection of line segments whose direction vectors are the elements of $V$. Zonotopes have a combinatorial decomposition that is useful when calculating volumes and counting lattice points. The following result is due to Shephard.

**Proposition 2.2** ([9, Theorem 54]). A zonotope $Z(V)$ can be subdivided into half-open parallelotopes that are in bijection with the linearly independent subsets of $V$.

A linearly independent subset $S \subseteq V$ corresponds under this bijection to the half-open parallelotope $\square S := \sum_{v \in S} (0, v]$. Stanley gave a combinatorial formula for the Ehrhart polynomial of a lattice zonotope.

**Theorem 2.3** ([10, Theorem 2.2]). Let $Z(V)$ be a lattice zonotope generated by $V$. Then

$$L_{Z(V)}(t) = \sum_{S \subseteq V \text{ lin. indep.}} \text{vol}(\square S) \cdot t^{|S|}.$$  \hspace{1cm} (3)

In the statement above and throughout the paper, volumes are normalized so that any primitive lattice parallelepiped has volume 1.

2.3 Fixed polytopes of the permutahedron

The symmetric group $S_n$ acts on $\mathbb{R}^n$ by permuting coordinates of points. The permutahedron $\Pi_n$ is the convex hull of the $S_n$-orbit of the point $(1, 2, \ldots, n) \in \mathbb{R}^n$; that is, of the $n!$ permutations of $[n]$.

Let $\sigma \in S_n$ be a permutation with cycles $\sigma_1, \ldots, \sigma_m$; their lengths form a partition $\lambda = (\ell_1, \ldots, \ell_m)$ of $n$. For each cycle $\sigma_k$ of $\sigma$, let $e_{\sigma_k} = \sum_{i \in \sigma_k} e_i$. The fixed polytope $\Pi_n^\sigma$ is defined to be the polytope consisting of all points in $\Pi_n$ that are fixed under the action of $\sigma$. We will use a few results from [2], which we now summarize.

**Theorem 2.4** ([2, Theorems 1.2 and 2.12]). The fixed polytope $\Pi_n^\sigma$ has the following zonotope description:

$$\Pi_n^\sigma = \sum_{1 \leq i < j \leq m} [\ell_i e_{\sigma_j} - \ell_j e_{\sigma_i}] + \sum_{k=1}^m \frac{\ell_k + 1}{2} e_{\sigma_k}. \hspace{1cm} (4)$$

Its normalized volume is

$$\text{vol } \Pi_n^\sigma = n^{m-2} \gcd(\ell_1, \ldots, \ell_m). \hspace{1cm} (5)$$

**Corollary 2.5.** The fixed polytope $\Pi_n^\sigma$ is integral or half-integral. It is a lattice polytope if and only if all cycles of $\sigma$ have odd length.

**Proof.** From (4) and from the fact that all of the $e_{\sigma_i}$ in (4) are linearly independent, we can see that all the vertices of $\Pi_n^\sigma$ will be in the integer lattice if and only if $\ell_i + 1$ is even for all $i$. \hfill \Box

Equation (4) also shows that $\Pi_n^\sigma$ is a rational translation of the zonotope $Z(V)$ where

$$V = \{\ell_i e_{\sigma_j} - \ell_j e_{\sigma_i} : 1 \leq i < j \leq m\}.$$ 

The following result characterizes the linearly independent subsets of $V$.

**Lemma 2.6** ([2, Lemma 3.2]). The linearly independent subsets of $V$ are in bijection with forests with vertex set $[m]$, where the vector $\ell_i e_{\sigma_j} - \ell_j e_{\sigma_i}$ corresponds to the edge connecting vertices $i$ and $j$. 


In light of this lemma, the fixed polytope $\Pi^n_\lambda$ gets subdivided into half-open parallelotopes $\square_F$ of the form

$$\square_F = \sum_{\{i,j\} \in E(F)} \ell_i e_{\sigma_j}, \ell_j e_{\sigma_i} + \sum_{k=1}^m \frac{\ell_k + 1}{2} e_{\sigma_k} + v_F, \quad v_F \in \mathbb{Z}^n$$

(6)

for each forest $F$ with vertex set $[m]$. When $F$ is a tree $T$ we have that $\text{vol}(\square_T) = \left(\prod_{j=1}^m \ell_j^{\deg_T(j)-1}\right) \text{gcd}(\ell_1, \ldots, \ell_m).$

by [2, Lemma 3.3]. For a general forest $F$, the parallelotopes $\square_T$ corresponding to each connected component $T$ of $F$ live in orthogonal subspaces, so

$$\text{vol}(\square_F) = \left(\prod_{j=1}^m \ell_j^{\deg_F(j)-1}\right) \left(\prod_{\text{conn. comp.} \, T \, \text{of} \, F} \text{gcd}(\ell_j : j \in \text{vert}(T))\right).$$

(7)

3 The Ehrhart polynomial of the fixed polytope: the lattice case

Suppose that $\lambda = (\ell_1, \ldots, \ell_m)$ is a partition of $n$ into odd parts and that $\sigma \in S_n$ has cycle type $\lambda$. Then Corollary 2.5 says that $\Pi^n_\lambda$ is a lattice zonotope, and hence we can use (3) to write a combinatorial expression for its Ehrhart polynomial. Recall the definition of $v_\pi$ in (1).

**Theorem 3.1.** Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$, where $\ell_i$ is odd for all $i$. Then

$$L_{\Pi^n_\lambda}(t) = \sum_{n=|\pi|} v_\pi \cdot t^{m-|\pi|}$$

summing over all partitions $\pi = \{B_1, \ldots, B_k\}$ of $[m]$.

**Proof.** Combining Theorem 2.3 with (7) gives us the following formula for the Ehrhart polynomial of $\Pi^n_\lambda$:

$$L_{\Pi^n_\lambda}(t) = \sum_{\text{Forests } F \text{ on } [m]} \left(\prod_{j=1}^m \ell_j^{\deg_F(j)-1}\right) \left(\prod_{\text{conn. comp.} \, T \, \text{of} \, F} \text{gcd}(\ell_j : j \in \text{vert}(T))\right) t^{|E(F)|}.$$

(8)

Note that we can construct a forest with vertex set $[m]$ by first partitioning $[m]$ into nonempty sets $\{B_1, \ldots, B_k\}$ and then choosing a tree with vertex set $B_j$ for each $j$. The number of edges in such a forest is $m - k$. Using these observations, we can rewrite (8) as

$$L_{\Pi^n_\lambda}(t) = \sum_{\{B_1, \ldots, B_k\} \subseteq [m]} \left(\prod_{i=1}^k \text{gcd}(\ell_j : j \in B_i)\right) \left(\sum_{\text{Forests } F \text{ inducing } \{B_1, \ldots, B_k\}} \prod_{j=1}^m \ell_j^{\deg_F(j)-1}\right) t^{m-k}.$$

To complete the proof, it remains to show that for a given partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$, the following identity holds:

$$\sum_{\text{Forests } F \text{ inducing } \{B_1, \ldots, B_k\}} \prod_{j=1}^m \ell_j^{\deg_F(j)-1} = \prod_{i=1}^k \left(\sum_{j \in B_i} \ell_j\right)^{|B_i| - 2}.$$

(9)

This follows from the following identity, found in [2, Lemma 3.4].

$$\sum_{T \text{ tree on } [m]} \prod_{i=1}^m x_i^{\deg_T(j)-1} = (x_1 + \cdots + x_m)^{m-2}.$$

(10)
Using (10) we obtain
\[
\sum_{\text{Forests } F \text{ inducing } \{B_1, \ldots, B_k\}} m \prod_{j=1}^{\deg F(j)-1} = \sum_{\text{Forests } F \text{ inducing } \{B_1, \ldots, B_k\}} \prod_{i=1}^{k} \prod_{j \in B_i} \deg F(j) - 1
\]
\[
= \prod_{i=1}^{k} \left( \sum_{T \text{ on } B_i} \deg T(j) - 1 \right)
\]
\[
= \prod_{i=1}^{k} \left( \sum_{j \in B_i} \ell_j |B_i| - 2 \right)
\]
as desired.

\[\square\]

4 The Ehrhart quasipolynomial of the fixed polytope: the general case

In general, \(\Pi_\sigma^n\) is a half-integral polytope. This means that instead of an Ehrhart polynomial, it has an Ehrhart quasipolynomial with period at most 2. As in the lattice case from Section 3, we can decompose \(\Pi_\sigma^n\) into half-open parallelotopes. However, there is a new feature that does not arise in the lattice case: some of the parallelotopes in this decomposition may not contain any lattice points.

Figure 2: Decomposition of the fixed polytope \(\Pi_{4}^{(12)}\) into half-open parallelotopes.

Example 4.1. The fixed polytope \(\Pi_{4}^{(12)}\) of Figure 1, which corresponds to the cycle type \(\lambda = (2,1,1)\), is
\[
\Pi_{4}^{(12)} = [2e_3, e_{12}] + [2e_4, e_{12}] + [e_4, e_3] + \frac{3}{2}e_{12} + e_3 + e_4.
\]

Figure 2 shows its decomposition into parallelograms indexed by the forests on vertex set \(\{12,3,4\}\). The three trees give parallelograms with volumes 2, 1, 1 that contain 2, 1, 1 lattice points, respectively. The three forests with one edge give edges of volumes 1, 1, 1 and 1, 1, 0 lattice points, respectively. The empty forest gives a point of volume 1 and 0 lattice points. Hence the Ehrhart quasipolynomial of \(\Pi_{4}^{(12)}\) is
\[
L_{\Pi_{4}^{(12)}}(t) = \begin{cases} 
(2 + 1 + 1)t^2 + (1 + 1 + 1)t + 1 & \text{if } t \text{ is even} \\
(2 + 1 + 1)t^2 + (1 + 1 + 0)t + 0 & \text{if } t \text{ is odd}
\end{cases}
\]

Following the reasoning of Example 4.1, we will find the Ehrhart quasipolynomial of \(\Pi_\sigma^n\) by examining its decomposition into half-open parallelotopes. In order to find the number of lattice points in each parallelotope \(\square_F\), the following observation is crucial.

Lemma 4.2. [1, 7] If \(\square\) is a lattice parallelotope in \(\mathbb{Z}^n\) and \(v \in \mathbb{Z}^n\), the number of lattice points in \(\square + v\) is
\[
|\square + v \cap \mathbb{Z}^n| = \begin{cases} 
\text{vol}(\square) & \text{if the affine span of } \square + v \text{ intersects the lattice } \mathbb{Z}^n \\
0 & \text{otherwise}
\end{cases}
\]
We now apply Lemma 4.2 to the parallelotopes $\Box_F$. Surprisingly, whether $\text{aff} (\Box_F)$ contains lattice points does not depend on the forest $F$, but only on the set partition $\pi$ of the vertex set $[m]$ induced by the connected components of $F$. To make this precise we need a definition. Recall that the 2-valuation of a positive integer is the largest power of 2 dividing that integer; for example, $\text{val}_2(24) = 3$.

**Definition 4.3.** Let $\lambda = (\ell_1, \ldots, \ell_m)$ be a partition of the integer $n$. A set partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$ is called $\lambda$-compatible if for each block $B_i \in \pi$, at least one of the following conditions holds:

(i) $\ell_j$ is odd for some $j \in B_i$, or

(ii) the minimum 2-valuation among $\{\ell_j : j \in B_i\}$ occurs an even number of times.

**Example 4.4.** Let $\lambda = (\ell_1, \ell_2, \ell_3)$ and $\text{val}_2(\ell_i) = v_i$ for $i = 1, 2, 3$, and assume that $v_1 \geq v_2 \geq v_3$. Table 1 shows which partitions of $[3]$ are $\lambda$-compatible depending on $\text{val}_2(\lambda)$.

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Table 1: $\lambda$-compatibility for $m = 3$.

**Lemma 4.5.** Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. Let $F$ be a forest on $[m]$ whose connected components induce the partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$. Then $\text{aff}(\Box_F)$ intersects the lattice $\mathbb{Z}^n$ if and only if $\pi$ is $\lambda$-compatible.

**Proof.** First we claim that

\[
\text{aff}(\Box_F) = \left\{ \sum_{j=1}^m x_j e_{\sigma_j} : \sum_{j \in B_i} \ell_j x_j = \sum_{j \in B_i} \frac{\ell_j (\ell_j + 1)}{2} \text{ for } 1 \leq i \leq k \right\}. \tag{11}
\]

Let $E(F)$ be the edge set of $F$. We have $\text{aff}(\Box_F) = \text{span}\{\ell_a e_{\sigma_a} - \ell_a e_{\sigma_a} : \{a, b\} \in E(F)\} + \sum_{a=1}^m \frac{1}{2}(\ell_a + 1)e_{\sigma_a}$. A point $y \in \text{span}\{\ell_a e_{\sigma_a} - \ell_a e_{\sigma_a} : \{a, b\} \in E(F)\}$ will satisfy $\sum_{j \in B_i} \ell_j y_j = 0$ for each block $B_i$. Furthermore, the translating vector $v := \sum_{a=1}^m \frac{1}{2}(\ell_a + 1)e_{\sigma_a}$ satisfies $\sum_{j \in B_i} \ell_j v_j = \sum_{j \in B_i} \frac{1}{2}\ell_j (\ell_j + 1)$ for each block $B_i$. Thus every point $x$ in the affine span of $\Box_F$ satisfies the given equations. These are all the relations among the $x_j$s because each block $B_i$ contributes $|E(B_i)| = |B_i| - 1$ to the dimension of the affine span of $\Box_F$.

This affine subspace intersects the lattice $\mathbb{Z}^n$ if and only if all equations in (11) have integer solutions. Elementary number theory tells us that this is the case if and only if each block $B_i$ satisfies

\[
\text{gcd}(\ell_j : j \in B_i) \left| \sum_{j \in B_i} \frac{\ell_j (\ell_j + 1)}{2} \right. \tag{12}
\]

It is always true that $\text{gcd}(\ell_j : j \in B_i)$ divides $\sum_{j \in B_i} \ell_j (\ell_j + 1)$, so (12) holds if and only if

\[
\text{val}_2 \left( \text{gcd}(\ell_j : j \in B_i) \right) < \text{val}_2 \left( \sum_{j \in B_i} \ell_j (\ell_j + 1) \right). \tag{13}
\]
We consider two cases.

(i) Suppose $\ell_j$ is odd for some $j \in B_1$. Then $\gcd(\ell_j : j \in B_1)$ is odd, whereas $\sum_{j \in B_1} \ell_j(\ell_j + 1)$ is always even. Hence (13) always holds in this case.

(ii) Suppose that $\ell_j$ is even for all $j \in B_1$. For each $\ell_j$, write $\ell_j = 2^{p_j}q_j$ for some integer $p_j \geq 1$ and odd integer $q_j$. Then $\text{val}_2(\gcd(\ell_j : j \in B_1)) = \min_{j \in B_1} p_j$; we will call this integer $p$. We have
\[
\text{val}_2 \left( \sum_{j \in B_1} \ell_j(\ell_j + 1) \right) = \text{val}_2 \left( \sum_{j \in B_1} 2^{p_j}q_j(\ell_j + 1) \right)
= p + \text{val}_2 \left( \sum_{j \in B_1} 2^{p_j}q_j(\ell_j + 1) \right).
\]
Note that $q_j(\ell_j + 1)$ is odd for each $j$. If the minimum 2-valuation $p$ of $\{\ell_j : j \in B_1\}$ occurs an odd number of times, then $\sum_{j \in B_1} 2^{p_j}q_j(\ell_j + 1)$ will be odd and we will have $\text{val}_2(\sum_{j \in B_1} \ell_j(\ell_j + 1)) = p$. Otherwise, this sum will be even and we will have $\text{val}_2(\sum_{j \in B_1} \ell_j(\ell_j + 1)) > p$. Therefore (13) holds if and only if the minimum 2-valuation among the $\ell_j$ for $j \in B_1$ occurs an even number of times. This is precisely the condition of $\lambda$-compatibility.

We now have all of the necessary tools to compute the Ehrhart quasipolynomial of the fixed polytope $\Pi_n^\sigma$. Recall the definition of $\lambda$-compatibility in 4.3 and the definition of $v_\pi$ in (1).

**Theorem 1.1.** Let $\sigma$ be a permutation of $[n]$ with cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. Then the Ehrhart quasipolynomial of the fixed polytope of the permutahedron $\Pi_n$ fixed by $\sigma$ is
\[
L_{\Pi_n^\sigma}(t) = \begin{cases} 
\sum_{\pi \in [m]} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is even} \\
\sum_{\pi \in [m]} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is odd}
\end{cases}_{\lambda-\text{compatible}}.
\]

**Proof.** We calculate the number of lattice points in each integer dilate $t\Pi_n^\sigma$ by decomposing it into half-open parallelotopes and adding up the number of lattice points inside of each parallelotope.

First, suppose that $t$ is even. Then $\Pi_n^\sigma$ is a lattice polytope, all parallelotopes in the decomposition of $t\Pi_n^\sigma$ have vertices on the integer lattice, and each $i$-dimensional parallelotope $\square$ contains $\text{vol}(\square)t^i$ lattice points [3, Lemma 9.2]. The parallelotopes correspond to linearly independent subsets of the vector configuration $\{e_{\sigma_i} - e_{\sigma_j} : 1 \leq i < j \leq m\}$, which is in bijection with forests on $[m]$. Following the reasoning used to prove Theorem 3.1, we conclude that when $t$ is even,
\[
L_{\Pi_n^\sigma}(t) = \sum_{\pi \in [m]} v_\pi \cdot t^{m-|\pi|}.
\]

Next, suppose $t$ is odd. Then $t\Pi_n^\sigma$ is half-integral, but it may not be a lattice polytope. As before, we may decompose $t\Pi_n^\sigma$ into half-open parallelotopes that are in bijection with forests on $[m]$. Lemma 4.2, Lemma 4.5, and [3, Lemma 9.2] tell us that $\square_F$ contains $\text{vol}(\square_F)t^{m-|\pi|}$ lattice points if the set partition $\pi$ induced by $F$ is $\lambda$-compatible, and 0 otherwise. Therefore if $t$ is odd
\[
L_{\Pi_n^\sigma}(t) = \sum_{\pi \in [m]} v_\pi \cdot t^{m-|\pi|}
\]
as desired. 
\qed
5 The equivariant \(H^*\)-series of the permutahedron

We now compute the equivariant \(H^*\)-series of the permutahedron and characterize when it is polynomial and when it is effective, proving Stapledon’s Effectiveness Conjecture 1.2 in this special case.

The \textit{Ehrhart series} of a rational polytope \(P\) is

\[
\text{Ehr}_P(z) = \sum_{t=0}^{\infty} L_P(t) \cdot z^t.
\]

In computing the Ehrhart series of \(\Pi_n^\sigma\), Eulerian polynomials naturally arise. The \textit{Eulerian polynomial} \(A_k(z)\) is defined by the identity

\[
\sum_{k \geq 0} A_k(z) \cdot z^k t^k = \frac{A_k(z)}{(1-z)^{k+1}}.
\]

Proposition 5.1. Let \(\sigma \in S_n\) have cycle type \(\lambda = (\ell_1, \ldots, \ell_m)\). The Ehrhart series of \(\Pi_n^\sigma\) is

\[
\text{Ehr}_{\Pi_n^\sigma}(z) = \sum_{\pi \in [m]} v_\pi \cdot A_{m-|\pi|}(z) \left(1 - z\right)^{m-|\pi|+1} + \sum_{\pi \in [m]} v_\pi \cdot 2^{m-|\pi|} \cdot A_{m-|\pi|}(z^2) \left(1 - z^2\right)^{m-|\pi|+1}
\]

and the \(H^*\)-series of the permutahedron equals

\[
H^*[z](\sigma) = \left( \prod_{i=1}^{m} (1 - z^{\ell_i}) \right) \cdot \text{Ehr}_{\Pi_n^\sigma}(z)
\]

Proof. The first statement follows readily from Theorem 1.1:

\[
\text{Ehr}_{\Pi_n^\sigma}(z) = \sum_{t \text{ even}} \left( \sum_{\pi \in [m]} v_\pi t^{m-|\pi|} \right) z^t + \sum_{t \text{ odd}} \left( \sum_{\pi \in [m]} v_\pi t^{m-|\pi|} \right) z^t
\]

\[
= \sum_{\pi \in [m]} v_\pi \left( \sum_{t=0}^{\infty} t^{m-|\pi|} z^t \right) + \sum_{\pi \in [m]} v_\pi \left( \sum_{t \text{ even}} t^{m-|\pi|} z^t \right)
\]

\[
= \sum_{\pi \in [m]} v_\pi \cdot A_{m-|\pi|}(z) \left(1 - z\right)^{m-|\pi|+1} + \sum_{\pi \in [m]} v_\pi \cdot 2^{m-|\pi|} \cdot A_{m-|\pi|}(z^2) \left(1 - z^2\right)^{m-|\pi|+1}
\]

For the second statement, recall that \(H^*[z]\) is defined as in (2), where \(\rho\) is the standard representation of \(S_n\) in this case. The left hand side is the Ehrhart series. The denominator on the right side is \((1-z) \text{det}(I-\rho(\sigma) \cdot z)\); it equals the characteristic polynomial of the permutation matrix of \(\sigma\), which is \(\prod_{i=1}^{m} (1-z^{\ell_i})\).

Tables 2 and 3 show the equivariant \(H^*\)-series of the permutahedra \(\Pi_3\) and \(\Pi_4\).

Stapledon writes that “The main open problem is to characterize when \(H^*[z]\) is effective”, and he conjectures the following characterization:

\textbf{Conjecture 1.2} ([11, Effectiveness Conjecture 12.1]). Let \(P\) be a lattice polytope invariant under the action of a group \(G\). The following conditions are equivalent.

(i) The toric variety of \(P\) admits a \(G\)-invariant non-degenerate hypersurface.

(ii) The equivariant \(H^*\)-series of \(P\) is effective.

(iii) The equivariant \(H^*\)-series of \(P\) is a polynomial.

He shows that (i) \(\implies\) (ii) \(\implies\) (iii), so only the reverse implications are conjectured. Our next goal is to verify Stapledon’s conjecture for the action of \(S_n\) on the permutahedron \(\Pi_n\). We do so by showing that the conditions of Conjecture 1.2 hold if and only if \(n \leq 3\).
Proof. First, suppose no pole at \( z \). Consider three cases:

- If all \( \lambda \) are even, the zero at \( z \) has cycle type \( \lambda \). Thus it follows that the order of the pole \( z \) has cycle type \( \lambda \). This equals \( \frac{1 + z + z^2}{1 - z} \), \( 1 + z^2 \).

- If all \( \lambda \) are odd, the zero at \( z \) has cycle type \( \lambda \). This equals \( \frac{1 + z + z^2}{1 - z^3} \), \( 1 + z + z^2 \).

- Otherwise, \( \lambda \) has a zero at \( z \) and \( \pi \) is incompatible, so \( \text{Ehr}_{\Pi_3}(z) \) has a pole at \( z = -1 \). Hence we only need to check whether \( H^*[z](\sigma) \) has a pole at \( z = -1 \).

(i) First, suppose no \( \ell_i \) is even. Then all partitions of \([m]\) are \( \lambda \)-compatible, so \( \text{Ehr}_{\Pi_3}(z) \) does not have a pole at \( z = -1 \). Thus \( H^*[z](\sigma) \) is a polynomial in this case.

(ii) Next, suppose that some \( \ell_i \) is even. Then the partition \( \{\{\ell_i\}, [m] - \{\ell_i\}\} \) is \( \lambda \)-incompatible, so \( \text{Ehr}_{\Pi_3}(z) \) does have a pole at \( z = -1 \). It is well known that \( A_k(1) = k! \) so every numerator \( v_{\pi} \cdot 2^{|\pi|} \cdot A_{m-|\pi|}(z^2) \) is positive at \( z = -1 \). It follows that the order of the pole \( z = -1 \) of \( \text{Ehr}_{\Pi_3}(z) \) is \( m - d + 1 \) where \( d = \min\{|\pi| : \pi \text{ is } \lambda \text{-incompatible}\} \). This equals \( m - 1 \) if the partition \( \{[m]\} \) is \( \lambda \)-compatible and \( m \) if it is \( \lambda \)-incompatible.

On the other hand, \( \prod_{i=1}^m (1 - z^{\ell_i}) \) has a zero at \( z = -1 \) of order equal to the number of even \( \ell_i \). Now consider three cases:

- If the number of even \( \ell_i \) is between 1 and \( m - 2 \), it is less than the order of the pole of \( \text{Ehr}_{\Pi_3}(z) \), so \( H^*[z](\sigma) \) is not polynomial.

- If all \( \ell_i \) are even, the zero \( z = -1 \) in \( \prod_{i=1}^m (1 - z^{\ell_i}) \) has order \( m \) and cancels the pole in \( \text{Ehr}_{\Pi_3}(z) \). Thus \( H^*[z](\sigma) \) is polynomial.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Cycle type of \( \sigma \in S_3 \) & \( \chi_{\Pi_3}(\sigma) \) & \( \sum_{t \geq 0} \chi_{\Pi_3}(\sigma) z^t \) & \( H^*[z](\sigma) \) \\
\hline
\((1, 1, 1)\) & \(3t^2 + 3t + 1\) & \(\frac{1 + 4z + z^2}{(1 - z)^3}\) & \(1 + 4z + z^2\) \\
\((2, 1)\) & \(\begin{cases} t + 1 & \text{if } t \text{ is even} \\ t & \text{if } t \text{ is odd} \end{cases}\) & \(\frac{1 + z^2}{(1 - z)(1 - z^2)}\) & \(1 + z^2\) \\
\((3)\) & 1 & \(\frac{1}{1 - z} = \frac{1 + z + z^2}{1 - z^3}\) & \(1 + z + z^2\) \\
\hline
\end{tabular}
\caption{The equivariant \( H^* \)-series of \( \Pi_3 \)}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Cycle type of \( \sigma \in S_4 \) & \( \chi_{\Pi_4}(\sigma) \) & \( \sum_{t \geq 0} \chi_{\Pi_4}(\sigma) z^t \) & \( H^*[z](\sigma) \) \\
\hline
\((1, 1, 1, 1)\) & \(16t^3 + 15t^2 + 6t + 1\) & \(\frac{1 + 34z + 55z^2 + 6z^3}{(1 - z)^4}\) & \(1 + 34z + 55z^2 + 6z^3\) \\
\((2, 1, 1)\) & \(\begin{cases} 4t^2 + 3t + 1 & \text{if } t \text{ is even} \\ 4t^2 + 2t & \text{if } t \text{ is odd} \end{cases}\) & \(\frac{1 + 6z + 20z^2 + 24z^3 + 11z^4 + 2z^5}{(1 - z)^5(1 - z^2)(1 + z)^2}\) & \(1 + 4z + 11z^2 - 2z^3 + \sum_{i=4}^{\infty} 4(-1)^i z^i\) \\
\((3, 1)\) & \(t + 1\) & \(\frac{1}{(1 - z)^2} = \frac{1 + z + z^2}{(1 - z)(1 - z^3)}\) & \(1 + z + z^2\) \\
\((4)\) & \(\begin{cases} 1 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}\) & \(\frac{1}{1 - z^2} = \frac{1 + z^2}{1 - z^4}\) & \(1 + z^2\) \\
\((2, 2)\) & \(\begin{cases} 2t + 1 & \text{if } t \text{ is even} \\ 2t & \text{if } t \text{ is odd} \end{cases}\) & \(\frac{1 + 2z + 3z^2 + 2z^3}{(1 - z)^2}\) & \(1 + 2z + 3z^2 + 2z^3\) \\
\hline
\end{tabular}
\caption{The equivariant \( H^* \)-series of \( \Pi_4 \)}
\end{table}

5.1 Polynomiality of \( H^*[z] \)

Lemma 5.2. Let \( \sigma \in S_n \) have cycle type \( \lambda = (\ell_1, \ldots, \ell_m) \). The equivariant \( H^* \)-series evaluated at \( \sigma \), \( H^*[z](\sigma) \), is a polynomial if and only if the number of even parts in \( \lambda \) is \( 0 \), \( m - 1 \), or \( m \).

Proof. By Proposition 5.1, the Ehrhart series \( \text{Ehr}_{\Pi_3}(z) \) may only have poles at \( z = \pm 1 \). The pole at \( z = 1 \) has order at most \( m \). Since the polynomial \( \prod_{i=1}^m (1 - z^{\ell_i}) \) has a zero at \( z = 1 \) of order \( m \), the series \( H^*[z](\sigma) \) will not have a pole at \( z = 1 \). Hence we only need to check whether \( H^*[z](\sigma) \) has a pole at \( z = -1 \).

(i) First, suppose no \( \ell_i \) is even. Then all partitions of \([m]\) are \( \lambda \)-compatible, so \( \text{Ehr}_{\Pi_3}(z) \) does not have a pole at \( z = -1 \). Thus \( H^*[z](\sigma) \) is a polynomial in this case.

(ii) Next, suppose that some \( \ell_i \) is even. Then the partition \( \{\{\ell_i\}, [m] - \{\ell_i\}\} \) is \( \lambda \)-incompatible, so \( \text{Ehr}_{\Pi_3}(z) \) does have a pole at \( z = -1 \). It is well known that \( A_k(1) = k! \) so every numerator \( v_{\pi} \cdot 2^{|\pi|} \cdot A_{m-|\pi|}(z^2) \) is positive at \( z = -1 \). It follows that the order of the pole \( z = -1 \) of \( \text{Ehr}_{\Pi_3}(z) \) is \( m - d + 1 \) where \( d = \min\{|\pi| : \pi \text{ is } \lambda \text{-incompatible}\} \). This equals \( m - 1 \) if the partition \( \{[m]\} \) is \( \lambda \)-compatible and \( m \) if it is \( \lambda \)-incompatible.

On the other hand, \( \prod_{i=1}^m (1 - z^{\ell_i}) \) has a zero at \( z = -1 \) of order equal to the number of even \( \ell_i \). Now consider three cases:

- If the number of even \( \ell_i \) is between 1 and \( m - 2 \), it is less than the the order of the pole of \( \text{Ehr}_{\Pi_3}(z) \), so \( H^*[z](\sigma) \) is not polynomial.

- If all \( \ell_i \) are even, the zero \( z = -1 \) in \( \prod_{i=1}^m (1 - z^{\ell_i}) \) has order \( m \) and cancels the pole in \( \text{Ehr}_{\Pi_3}(z) \). Thus \( H^*[z](\sigma) \) is polynomial.
When $m$ and $n$ are even, the partition $\{m\}$ is $\lambda$-compatible. Therefore the order of the pole in $Ehr_{\Pi_n}(z)$ and the order of the zero in $\prod_{i=1}^{m}(1-z^{\ell_i})$ both equal $m-1$, and $H^\ast[z](\sigma)$ is polynomial.

**Proposition 5.3.** The equivariant $H^\ast$-series of the permutahedron $\Pi_n$ is a polynomial if and only if $n \leq 3$.

**Proof.** When $n \leq 3$, all partitions of $n$ have 0, 1, or all odd parts. Hence $H^\ast[z](\sigma)$ is a polynomial for all $\sigma \in S_n$, so $H^\ast[z]$ is a polynomial.

Suppose $n \geq 4$. Then there always exists some partition of $n$ with more than 1 but fewer than all odd parts: if $n$ is even we can take the partition $(n-2,1,1)$, and if $n$ is odd we can take the partition $(n-3,1,1,1)$. Therefore $H^\ast[z]$ is not polynomial.

\[ \therefore \]

**5.2 Effectiveness of $H^\ast[z]$**

**Proposition 5.4.** The equivariant $H^\ast$-series of the permutahedron $\Pi_n$ is effective if and only if $n \leq 3$.

**Proof.** Stapledon [11] observed that if $H^\ast$ is effective then it is a polynomial. Thus by Proposition 5.3 we only need to check effectiveness for $n = 1, 2, 3$.

Let us check it for $n = 3$. Table 2 shows that $H^\ast[z] = H^\ast_0 + H^\ast_1 z + H^\ast_2 z^2$ for $H^\ast_0, H^\ast_1, H^\ast_2 \in R(S_3)$. Comparing these with the character table of $S_3$ (see for example [5, pg.14]) gives

\[
H^\ast_0 = \chi_{\text{triv}}, \quad H^\ast_1 = \chi_{\text{triv}} + \chi_{\text{alt}} + \chi_{\text{std}}, \quad H^\ast_2 = \chi_{\text{triv}}.
\]

Since all coefficients are nonnegative, $H^\ast_{\Pi_3}[z] = \chi_{\text{triv}} + (\chi_{\text{triv}} + \chi_{\text{alt}} + \chi_{\text{std}})z + \chi_{\text{triv}}z^2$ is indeed effective.

Similarly, $H^\ast_{\Pi_2}[z] = \chi_{\text{triv}}$ and $H^\ast_{\Pi_1}[z] = \chi_{\text{triv}}$ are effective as well.

In contrast, a similar computation based on Table 3 gives

\[
H^\ast_{\Pi_1} = \chi_{\text{triv}} + (3\chi_{\text{triv}} + \chi_{\text{alt}} + 5\chi_{\text{std}} + 3\chi_{\text{std}} + \chi_{\text{std}} + 6\chi_{\text{std}} + 9\chi_{\text{std}} + 4\chi_{\text{std}} + 5\chi_{\text{std}})z^2
\]

\[
+ (\chi_{\text{alt}} + \chi_{\text{std}} + \chi_{\text{std}})z^3 + (\chi_{\text{triv}} - \chi_{\text{alt}} + \chi_{\text{std}} - \chi_{\text{std}})(z^4 - z^5 + z^6 - z^7 + \cdots)
\]

which is not effective.

\[ \therefore \]

**5.3 $S_n$-invariant non-degenerate hypersurfaces in the permutahedral variety**

We begin by explaining condition (i) of Conjecture 1.2, which arises from Khovanskii’s notion of non-degeneracy [6]. We refer the reader to [11, Section 7] for more details.

Let $P \subset \mathbb{R}^n$ be a lattice polytope with an action of a finite group $G$. For $v \in \mathbb{Z}^n$ we write $x^v := x_1^{v_1} \cdots x_n^{v_n}$. The coordinate ring of the projective toric variety $X_P$ of $P$ has the form $\mathbb{C}[x^v : v \in P \cap \mathbb{Z}^n]$, so a hypersurface in $X_P$ is given by a linear equation $\sum_{v \in P \cap \mathbb{Z}^n} a_v x^v = 0$ for some complex coefficients $a_v$. The group $G$ acts on the monomials $x^v$ by its action on the lattice points $v \in P \cap \mathbb{Z}^n$, so the equation of a $G$-invariant hypersurface should have $a_v = a_w$ whenever $u$ and $v$ are in the same $G$-orbit. A projective hypersurface in $X_P$ with equation $f(x_1, \ldots, x_n) = 0$ is smooth if the gradient $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is never zero when $(x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$. There is a unique polynomial in the $a_v$, called the discriminant, such that the hypersurface is smooth when the discriminant does not vanish at the coefficients $a_v$. A hypersurface in the toric variety of $P$ is non-degenerate if it is smooth and for each face $F$ of $P$, the hypersurface $\sum_{v \in F \cap \mathbb{Z}^n} a_v x^v = 0$ is also smooth.

The permutahedral variety $X_{\Pi_n}$ is the projective toric variety associated to the permutahedron $\Pi_n$.

**Proposition 5.5.** The permutahedral variety $X_{\Pi_n}$ admits an $S_n$-invariant non-degenerate hypersurface if and only if $n \leq 3$.

**Proof.** Stapledon proved [12, Theorem 7.7] that if $X_{\Pi_n}$ admits such a hypersurface, then $H^\ast[z]$ is effective. By Proposition 5.4, this can only occur for $n = 1, 2, 3$.

Case 1: $n = 1$.

A hypersurface in the toric variety of $\Pi_1 = \{1\} \subset \mathbb{R}$ has the form $ax = 0$, and since we are working over projective space, we can assume $a = 1$. The derivative of this never vanishes, so this is a smooth $S_1$-invariant hypersurface.
Case 2: $n = 2$.

The permutohedron $\Pi_2$ is the line segment with vertices $(1,2), (2,1) \in \mathbb{R}^2$ and no other lattice points. The vertices are in the same $S_2$-orbit, so we need to check that hypersurface with equation $xy^2 + x^2y = 0$ is non-degenerate. The gradient is $(y(y+2x), x(2y+x))$, which never vanishes on $(\mathbb{C}^*)^2$. The vertex $(1,2)$ corresponds to the hypersurface $xy^2 = 0$. The gradient of this is $(y^2, 2xy)$ which also never vanishes on $(\mathbb{C}^*)^2$. The computation for the other vertex is similar. Hence this is an $S_2$-invariant non-degenerate hypersurface.

Case 3: $n = 3$.

The permutohedron $\Pi_3$ is a hexagon with one interior point. Choosing the vertices to be all permutations of the point $(0,1,2) \in \mathbb{R}^3$ (instead of $(1,2,3)$) will simplify calculations. The six vertices of the hexagon are one $S_3$-orbit and the interior point is its own orbit. Hence (up to scaling) an $S_3$-invariant hypersurface must have the equation

$$a \cdot xyz + yz^2 + z^2y + x^2y + xz^2 + x^2z = 0$$

which has one parameter $a$. We want to check whether there exists some choice of $a$ for which this hypersurface is non-degenerate. We need to check this on each face.

The vertex $(0,1,2)$ gives the hypersurface $yz^2 = 0$ with gradient $(0,z^2,2yz)$. This never vanishes on $(\mathbb{C}^*)^3$, so it is smooth. The computations for the other five vertices are similar.

For the edge connecting $(0,1,2)$ and $(0,2,1)$, the corresponding hypersurface is $yz^2 + y^2z = 0$. This is the same hypersurface as the line segment $\Pi_2$, so it is smooth; so are the hypersurfaces of the other five edges.

Finally, we need to show there exists a such that the entire hypersurface is smooth. This is the same as showing that the discriminant of (14) is not identically zero. Since (14) is a symmetric polynomial, we can write in terms of the power-sum symmetric polynomials, $p_k = x^k + y^k + z^k$; we obtain

$$\frac{a}{6}p_1^3 + \left(1 - \frac{a}{2}\right)p_1p_2 + \left(\frac{a}{3} - 1\right)p_3 = 0.$$  

(15)

The discriminant of a degree 3 symmetric polynomial is given in [8, Equation 64]; substituting the coefficients $a/6, 1 - a/2, a/3 - 1$ gives a non-zero polynomial of degree 12:

$$\frac{-512000}{1667718169996569}a^{12} + \frac{492800}{25290}a^{10} - \frac{985600}{27411}a^8 - \frac{620}{282429536481}a^6 + \frac{9054207}{3138109969}a^4 + \frac{282429536481}{3138109969}a^2 + \frac{282429536481}{3138109969}a^2$$

Any value of $a$ that is not a root of this discriminant gives us an $S_3$-invariant non-degenerate hypersurface. 

By contrast, we should not be able to find an $S_n$-invariant non-degenerate hypersurface in $X_{\Pi_n}$ for $n \geq 4$. This can be seen from the fact that all permutohedra $\Pi_n$, when $n \geq 4$ have a square face, and the hypersurface of this square face is not smooth. For example, consider the square face of $\Pi_4$ with vertices $(0,1,2,3), (0,1,3,2), (1,0,3,2), \text{and } (1,0,2,3)$. The corresponding hypersurface is $yz^2w^3 + yz^3w^2 + xz^3w^2 + xz^2w^3 = 0$, and its gradient vanishes whenever $x = -y$ and $z = -w$.

5.4 Stapledon’s Conjectures

Our second main result now follows as a corollary.

**Theorem 1.3.** Stapledon’s Effectiveness Conjecture holds for the permutohedron under the action of the symmetric group.

**Proof.** This follows immediately from Propositions 5.3, 5.4, and 5.5. 

In closing, we verify the remaining three conjectures of Stapledon for the special case of the $S_n$–action on the permutohedron $\Pi_n$.

**Conjecture 5.6.** [11, Conjecture 12.2] If $H^*[z]$ is effective, then $H^*[1]$ is a permutation representation.
Conjecture 5.7. [11, Conjecture 12.3] For a polytope $P \subset \mathbb{R}^n$, let $\text{ind}(P)$ be the smallest positive integer $k$ such that the affine span of $kP$ contains a lattice point. For any $g \in G$, let $M^g$ be the sublattice of $M$ fixed by $g$, and define $\det(I - \rho(g))_{\cdot(M^g)^\perp}$ to be the determinant of $I - \rho(g)$ when the action of $\rho(g)$ is restricted to $(M^g)^\perp$. The quantity
\[
H^*[1](g) = \frac{\dim(P^g)! \cdot \text{vol}(P^g) \cdot \det(I - \rho(g))_{\cdot(M^g)^\perp}}{\text{ind}(P^g)}
\]
is a non-negative integer.

Conjecture 5.8. [11, Conjecture 12.4] If $H^*[z]$ is a polynomial and the $i$th coefficient of the $h^*$-polynomial of $P$ is positive, then the trivial representation occurs with non-zero multiplicity in the virtual character $H^*_i$.

Proposition 5.9. Conjectures 5.6, 5.7, and 5.8 hold for permutahedra under the action of the symmetric group.

Proof. 5.6: This statement only applies to $\Pi_1$, $\Pi_2$, and $\Pi_3$. From the proof of Proposition 5.4 we obtain that $H^*[1]$ is the trivial character for $\Pi_1$ and $\Pi_2$ and the statement holds. For $\Pi_3$ we have
\[
H^*[1] = 3\chi_{\text{triv}} + \chi_{\text{alt}} + \chi_{\text{std}} = \chi_{\text{triv}} + (\chi_{\text{triv}} + \chi_{\text{alt}}) + (\chi_{\text{triv}} + \chi_{\text{std}}).
\]
Now $\chi_{\text{triv}} + \chi_{\text{alt}}$ is the permutation character of the sign action of $S_3$ on the set $\{2\}$, and $\chi_{\text{triv}} + \chi_{\text{std}}$ is the character of the permutation representation of $S_3$. Hence all summands on the right side of (16) are permutation characters, so their sum is as well.

5.7: For $\sigma \in S_n$ of cycle type $\lambda = (\ell_1, \ldots, \ell_m)$ we have $\dim(\Pi_n^\sigma) = m-1$ and $\text{vol}(\Pi_n^\sigma) = m^{m-2} \gcd(\ell_1, \ldots, \ell_m)$. Now, the fixed lattice $M^g = \mathbb{Z}\{e_{\sigma_1}, \ldots, e_{\ell_m}\}$ has rank $m$, so
\[
\det(I - \rho(\sigma) \cdot z)_{\cdot(M^g)^\perp} = \frac{(1 - z) \det(I - \rho(\sigma) \cdot z)}{(1 - z)^m} = \prod_{i=1}^m (1 + z + \cdots + z^{\ell_i - 1}).
\]
Therefore the numerator is $(m - 1)! \cdot n^{m-2} \cdot \gcd(\ell_1, \ldots, \ell_m) \cdot \ell_1 \cdots \ell_m$. The denominator is
\[
\text{ind}(\Pi_n^\sigma) = \begin{cases} 
2 & \text{if all } \pi \vdash [m] \text{ are } \lambda\text{-incompatible}, \\
1 & \text{otherwise.} 
\end{cases}
\]
When the denominator is 2, all the $\ell_i$ must be even, so the numerator is even. The desired result follows.

5.8: We need to check this for $\Pi_1$, $\Pi_2$, and $\Pi_3$. For $\Pi_1$ and $\Pi_2$ the $h^*$-polynomial is 1 and $H^*_0 = \chi_{\text{triv}}$. For $\Pi_3$, the $h^*$-polynomial is $1 + 4z + z^2$, and $H^*_0 = \chi_{\text{triv}}$, $H^*_1 = \chi_{\text{triv}} + \chi_{\text{alt}} + \chi_{\text{std}}$, and $H^*_2 = \chi_{\text{triv}}$ all contain a copy of the trivial character. \qed

6 Acknowledgments

We would like to thank Sophia Elia for developing a useful Sage package for equivariant Ehrhart theory, and Matthias Beck, Benjamin Braun, Christopher Borger, Ana Botero, Sophia Elia, Donghyun Kim, Jodi McWhirter, Dusty Ross, Kristin Shaw, and Anna Schindler for fruitful conversations. This work was completed while FA was a Spring 2019 Visiting Professor at the Simons Institute for Theoretical Computer Science in Berkeley, and a 2019-2020 Simons Fellow while on sabbatical in Bogotá; he is very grateful to the Simons Foundation, San Francisco State University, and the Universidad de Los Andes for their support.

References


