Probability and Statistics
Comprehensive Masters Exam - June 2005

INSTRUCTIONS:

You are allowed two and half hours.
Choose any eight questions.

1. A box contains five coins. Let p_i denote the probability of a head when the ith coin is tossed ($i = 1,2,3,4,5$) and suppose that $p_i = \frac{i-1}{4}$. Suppose that one coin is selected at random from the box and when it is tossed once, a head is obtained. What is the probability that the ith coin was selected ($i = 1,2,3,4,5$)?

2. Let the random variable X have a pdf defined by $f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$. Let the random variable $Y = \frac{1}{X}$. Find the pdf of Y and use it to compute $E(Y)$.

3. Let X be a random variable such that $E(X^n) = (n+1)!2^n$, $n = 1,2,3\ldots$. Show that X has a chi-squared distribution with 4 degrees of freedom.

4. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from a distribution that is $N(\mu, \sigma^2)$. Show that the random variable $Y = \sum(\frac{X_i - \mu}{\sigma})^2$ has a chi-squared distribution with n degrees of freedom.

5. Complete the following:

a) Use the definition of the moment generating function to find the moment-generating function for a Poisson random variable.

b) Suppose X_1, X_2, X_3 are mutually independent Poisson random variables with means 2, 1, 4 respectively.

i) Find the moment generating function for $Y = X_1 + X_2 + X_3$.

ii) Find $P(Y = 5)$

6. Suppose $f_{X,Y}(x,y) = \frac{3(4-2x-y)}{16}$ for $x > 0$, $y > 0$, $2x + y < 4$, find $P(Y > 2|X = 1/2)$.

7. Let $f_{X,Y}(x,y) = \frac{1}{8}$ for $-2 < x < 2$, $0 < y < 2$. Find $f(z)$ where $Z = X + Y$.

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8. Suppose scores on a standardized mathematics test taken by students from large and small high schools are summarized in the table below.

	Large HS	Small HS
Sample size	9	12
Sample mean	81.3	78.6
Sample standard deviation	7.8	6.9

For each test below, state the name of the test, the assumptions required to perform the test, the hypotheses you are testing,, the significance level, and your conclusion.

a) Test whether there is evidence the true variances of the scores from large and small high schools are equal or not?

b) Test whether there is evidence that students from large high schools do better on this test than students from small high schools.

9. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from a distribution that is $N(\mu, \sigma^2)$. Show that $\overline{X}$ is

a) the maximum-likelihood estimator for μ.

b) unbiased for μ.

c) efficient for μ.

d) consistent for μ.

e) Moreover, if σ^2 is known, $\overline{X}$ is sufficient for μ.

10. Prove that. for any two random variables X and Y,

a) $|\rho(X, Y)| \leq 1$

b) $|\rho(X, Y)| = 1$ if and only if $Y = aX + b$ for some constants a and b.