MA Culminating Written Exam
Syllabus of Topics

February 8, 2005

1 Summary

This document gives a detailed list of topics that the graduate student taking the algebra exam is expected to know. The topics here are drawn from four courses: Math 325 Linear Algebra, Math 335 Modern Algebra, Math 725 Advanced Linear Algebra, and Math 850 Graduate Algebra. The material presented here reflects an approximation to a common denominator of the topics covered by various instructors in these courses over the last 2-3 years. The syllabus is grouped into four categories: Group Theory, Ring Theory, Modules and Fields, and Linear Algebra. A bibliography can be found at the end of the syllabus. For each topic mentioned there will be references to where the student can study it in the various books. The student is highly encouraged to look at and try to solve exercises related to the topics in the sources referenced.

2 Syllabus

2.1 Group Theory

Groups: Basic Properties. Binary operations (examples and non-examples), definition of a group, simple examples such as $\mathbb{Z}_n$, basic properties (uniqueness of inverses, exponent laws, etc.) [3, §1.1], [4, Ch. 2], [5, §2.1-2.2], [6, §2.1-2.3], [8, Ch. 3-4].

Examples. Dihedral groups [3, §1.2], [4, Ch. 1]; Matrix groups [3, §1.4], [4, Ch. 2].

Permutation/Symmetric Groups. Definition, cycle notation, generators, transpositions, even/odd permutations [3, §1.3, §3.5], [4, Ch. 5], [5, Ch. 3], [6, §2.10], [8, Ch. 7-8].

Subgroups. Definition, recognizing a subgroup, examples such as normalizers, centralizers, stabilizers, subgroups generated by subsets [3, §2.1-2.2 §2.4], [4, Ch. 3], [5, Ch. 3], [8, Ch. 5].

Cyclic Groups. Definition, order of a group element and related properties, finite and infinite cyclic groups, classification of subgroups of cyclic groups, distinct generators of a cyclic group [3, §2.3], [4, Ch. 4], [5, §2.4], [8, Ch. 10-11].

Homomorphisms, Isomorphisms, Normal Subgroups. Group homomophisms and isomorphisms, simple examples, image and kernel of homomorphisms, definition of normal subgroups, *Theorem*: kernels are normal subgroups and vice versa, Cayley's Theorem [3, §1.6, §3.1], [4, Ch. 6, 9, 10], [5, Ch. 2.5], [8, Ch. 9,14].

Cosets, Factor/Quotient Groups, Lagrange's Theorem. Left and right cosets, Lagrange's theorem, groups of prime order, Fermat's Little Theorem, factor/quotient groups, isomorphism theorems [3, §3.2-3.3],[4, Ch. 7,9,10],[5, §2.4, §2.6-2.7], [6, §2.6-2.7], [8, Ch. 13, 15-16].

Group Actions, Conjugacy, and Class Equation. Definition of group actions and examples, orbits, conjugation action, conjugacy class, conjugacy classes in the symmetric group, class equation and its consequences [3, §4.1-4.3],[4, Ch. 24],[5, §2.11], [6, §2.11], [8, Ch. 16].

Direct Products and Abelian Groups. Definition of direct product, basic properties, and examples, *statement* of the Fundamental Theorem of Finitely Generated Abelian Groups, classification examples of finite abelian groups [3, §5.1-5.3],[4, Ch. 11],[5, §2.9-2.10], [8, Ch. 16].

2.2 Ring Theory

Rings: Elementary Properties and Examples. Definition of rings, unit, zero-divisor, division ring, integral domain, field; basic properties; examples: integers, polynomial rings, matrix rings, power series rings, group rings [3, §7.1-7.2],[4, Ch. 12-13],[5, §4.1-4.2], [6, §3.1-3.2], [8, Ch. 17].

Homomorphisms, Ideals, and Quotient/Factor Rings. Ring homomophisms and isomorphisms, subrings, images and kernels, definition of ideals and quotient rings, one- and two-sided ideals, isomorphism theorems, properties of ideals, maximal and prime ideals, rings of fractions [3, §7.3-7.5],[4, Ch. 14-15],[5, §4.3-4.4, §4.7], [6, §3.3-3.6], [8, Ch. 18-19].

PIDs, UFDs, and Polynomial Rings. Definition of PID, greatest common divisor, prime and irreducible elements, ring of integers, unique factorization, definition and examples of UFD, PIDs are UFDs, basic properties of polynomial rings, polynomial rings over fields, division algorithm, Gauss' Lemma, irreducibility of polynomials, Eisenstein's criterion [3, §8.2-8.3, §9.1-9.4],[4, Ch. 16-18],[5, §4.5-4.6], [6, §3.9-3.11], [8, Ch. 21-22, 24-26].

2.3 Modules and Fields

Modules: Definition and Examples. Definition of modules, abelian groups as $\mathbb{Z}$-modules, vector spaces as F-modules, vector spaces with linear transformations as $F[x]$-modules, submodules [3, §10.1], [6, §4.5], [8, Ch. 21-22, 24-26];
module homomorphisms, quotient modules, isomorphism theorems, direct sums, free modules [3, §10.2-10.3].

Fields, Field Extensions, and Constructions. Characteristic of a field, field extension, degree of an extension, computing in a finite extension, algebraic extension, minimal polynomial, roots of unity, ruler and compass constructions [3, §13.1-13.3],[4, Ch. 20-21, 23],[5, §5.3-5.5], [6, §3.1-3.4], [8, Ch. 27, 29-30].

2.4 Linear Algebra

All the vector spaces treated here are finite-dimensional, and most of them are vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

Vector Spaces: Definitions and Elementary Properties. Definition of vector spaces, examples, basic properties, subspaces, direct sum, quotient vector space, linear independence, bases, dimension [1, Ch. 1-2], [2, §3.2-3.3, §4.1], [3, §11.1], [6, §4.1-4.2].

Linear Maps. Definition and examples of linear maps, kernel (nullspace) and image (range) of a linear map, rank-nullity theorem, the matrix of a linear map, matrix multiplication, invertibility of a map (matrix), $\mathrm{Hom}(V, W)$ and its properties. [1, Ch. 3], [2, Ch. 2], [3, §11.2], [6, §4.3, 6.1, 6.3].

Eigenvalues and Eigenvectors. Invariant subspaces, eigenvalues and eigenvectors of a linear map, eigenbases, diagonalization, real eigenvalues, generalized eigenvectors, nilpotent operators, characteristic polynomial, Cayley-Hamilton theorem, minimal polynomial, Jordan form [1, Ch. 5,8], [2, §7.2-7.5], [3, §12.3], [6, §6.2, 6.6].

Inner Product Spaces. Inner product and norm, orthonormal bases, Gram-Schmidt process, linear functionals and adjoints, self-adjoint and normal operators, real and complex spectral theorems, positive operators, singular value decomposition [1, Ch. 6,7], [2, §5.1-5.5, §8.3], [6, §6.10].

References

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[2] Otto Bretcher. *Linear Algebra with Application, Third Edition* 2005 Prentice Hall.

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