Measuring Symmetries activity

This lesson is designed to engage students with the mathematical concept of symmetry through guided discovery. This geometric and algebraic concept introduces students to the idea of “mathematics without numbers”, showing students that the world of mathematics is not limited to arithmetic and solving simple equations.

Even when a math class is focused on developing every student’s abilities in an equitable way, it seems that many exercises do not engage students’ creativity. Many problems, even problems considered group-worthy, are solved by simply applying an algorithm to a given data set. I have seen few activities at the secondary level that succeed in engaging students’ creativity. This lesson depends entirely on students’ creative mathematical ideas.

Also, math students tend to forget mathematical ideas as quickly as—and often more quickly than—they learn them. This is normal. Exposure to mathematical ideas through definitions and formulas teaches students to quickly learn algorithms, apply them on homework assignments and tests, then set them aside. Students truly learn mathematics when they understand why an algorithm works and its necessity in a larger conceptual framework. This is a framework each student must build for herself, a teacher cannot do it for her. This lesson depends on students to generate mathematics themselves, in hopes that any mathematical understanding will also be personal, therefore lasting, understanding.

Motivation

As an undergraduate, I participated in a research project, led by Portland State University Assistant Professor Sean Larsen, dedicated to creating an undergraduate introductory abstract algebra curriculum based on guided discovery. I was first exposed to the curriculum as a research subject, and again as a student in PSU’s group theory course. As a research subject,
another student and I met weekly with Dr. Larsen—with no book, no assigned reading, and no homework. We developed definitions for and the basic properties of algebraic groups, subgroups, and quotient groups via an exploration of the symmetries of an equilateral triangle. Each week I was discovering deep, exciting mathematics for myself rather than being given a set of axioms and being told to prove a series of provided theorems. This experience remains one of the highlights of my mathematical education.

Of course, high school students should not be expected to understand abstract algebra—they are busy learning to understand basic algebra’s abstraction of problems from life. The idea here is to use some somewhat “concrete” objects of geometry and design along with the aesthetically familiar concept of symmetry to provide students with an opportunity to see math as a domain of investigation, rather than a tool for solving problems.

Mathematical background

Examples of symmetry abound in both nature and design.

Each of the above examples exhibits the bilateral symmetry that exemplifies “symmetry” as understood by the general public. This type of symmetry is called reflection. Notice that the snowflake has more than one line of reflection, unlike the other examples. So in some way, we can view the snowflake as being “more symmetric” than the leaf, the Taj Mahal, or Mickey. The snowflake has another type of symmetry lacking in the other examples. It can be picked up, rotated by any multiple of 60 degrees, then put back down without altering its appearance. This is called rotational symmetry.

A design (see next page) by M. C. Escher uses another type of symmetry. Escher slides a basic design element vertically and horizontally to fill the plane, providing an example of the symmetry of translation.

Not all patterns may reflected, rotated, or translated in the same way. Of our examples, only the snowflake and Escher’s design have rotational
symmetry (can you see Escher’s point of rotation?). Escher’s print cannot be reflected across any line, unlike our other examples. The symmetries of different objects and patterns are formalized mathematically in abstract algebra as symmetry groups.

The basic notion of abstract algebra is that of a group, which is a set, $G$, together with an operation on the set, $\star$, satisfying three axioms.

i. The operation is associative, that is

\[(a \star b) \star c = a \star (b \star c)\]

for all elements $a$, $b$, and $c$ in $G$.

ii. The set contains an identity element, that is there is some element $e$ in $G$ such that

\[a \star e = e \star a = a\]

for all $a$ in $G$.

iii. Each element of the set has an inverse element, that is, given some element $a$, there exists some element $b$ such that

\[a \star b = b \star a = e\]

where $e$ is the identity element of $G$. 
Familiar examples of groups include the set of integers with the operation addition and the set of rational numbers with the operation multiplication. (Quick quiz: Are the integers still a group with the operation multiplication?) All the results of abstract algebra follow from this definition.

Now consider a set consisting of those symmetries (called isometries in formal mathematics) which leave a figure in the plane unchanged, viewed as functions. The operation is composition of these functions. For example, the isometries for an equilateral triangle are three rotations (of 0°, 120°, and 240°) and three reflections (across each bisected angle). The 0° rotation serves as the group’s identity element. Each reflection is its own inverse element, and the non-trivial rotations serve as each other’s inverse elements. Playing with an equilateral triangle quickly “convinces” one that the operation is associative. Thus these isometries form a group, called a symmetry group. Symmetry groups are one of the basic objects of study in group theory. Different objects can be associated with same symmetry group, leading to classification of symmetry types. Our lesson will engage students in classifying geometric patterns by symmetry type, based on their own classification schemes.

Students’ background

Students have been exposed to symmetry in nature and design, and have developed their intuitive understandings of symmetry. Of course, we do not expect students to have experience with the mathematical concept of symmetry. They do, however, have experience with polygons and associated algebraic and geometric concepts like area and perimeter. In my experience in a ninth grade classroom, I have observed that for many students the concepts of perimeter and area are not clear geometrically (as definitions) or algebraically (as formulas). These are examples of concepts most students have been told repeatedly. This lesson will encourage students to discover a method of measurement themselves. This will give them some ownership of the idea of symmetry.

Students’ understanding of mathematical discourse at this point appears to be of the “this is the answer” variety. This lesson will demand more from them, as there is no correct answer to many of the questions.
Lesson plan

[This activity is adapted from Sean Larsen’s materials.]

This lesson’s goal is for students to discover that the idea of a symmetry leads to the idea of symmetry types, that is, that the symmetries associated with different geometric objects can be essentially the same. Students will rank geometric objects by symmetry, and be asked to find a method to measure the symmetry of a given object. To do this, each group must devise a method of quantifying the symmetry of a figure using intuitive ideas of what makes one figure more or less symmetric than another. Students will share posters and justification for their quantification scheme. Which symmetries should be considered “the same”?

The mathematical themes of the lesson are justification, equivalence, student thinking, and algebraic structure.

- Justification — Groups must justify their quantification scheme so that it aligns with their intuitive understanding of symmetry.

- Equivalence — Which symmetries do students consider equivalent? Why?

- Student thinking — There is no right answer here; students should be encouraged to listen carefully to each other’s thinking during the activity.

- Algebraic structures — The activity is designed to prepare students for a careful examination of a single symmetry group (given another class period, it would be good to explore the symmetries of one figure in detail, leading to the basic properties of an algebraic group).

The goals of the activity are to

- promote a safe, supportive, challenging classroom environment;

- highlight the role of student thinking in teaching and learning;

- develop students’ understandings of symmetry and equivalence; and

- make connections between algebra and geometry.
The lesson as given to students is the last page of this paper. Parts one and two give students a chance to share their thinking and build on their individual understanding of symmetry. Parts three through five are designed to provoke discussion and thinking about symmetry and equivalence by having students come to group consensus and connect algebra and geometry. Making and sharing posters encourages students to synthesize their learning and question and challenge each other’s thinking.

Here are a few questions to stimulate student discussions:

- When are two symmetries the same?
- Regarding Figure F, is a 90 degree rotation more like a 450 degree rotation or more like a 180 degree rotation?
- Should no rotation at all count as a symmetry? What about a 360 degree rotation?
- What criteria should be used to determine if two symmetries are the same? Is this a general enough criteria to apply to any figure?

Reflection on lesson

I led this activity in two ninth grade algebra classes on Thursday December 17th. This was the final day of class before winter break. Students had taken their semester final the previous class, and this activity was presented as a fun last class activity before vacation.

The first thing I need to reflect on is my failure to include any set-up for this activity in the lesson plan. Before starting this activity, the instructor must be sure that every student has a good idea was is meant by “symmetry”. I used a Do Now activity asking if objects were symmetric or not and why. This got students thinking about reflection symmetry. I hinted during the Do Now discussion that rotations could be considered symmetries, but left discovery of this symmetry type to student groups. The questions that best sparked discussion of rotational symmetry were “Why did you rank object G as less symmetric than Figures E and F?” and “Why is E more (or less—depending on the group) symmetric than F?” Another set-up issue is to provide a reason for ranking these figures. Why do we care if one figure is more or less symmetric than another?

The structure of the activity demanded lots of instructor activity. It was a challenge to keep “private think time” private for everyone. Also, part
two of the activity needs to be carefully managed to guarantee that every student shares his or her ranking. (I could have done a better job here.)

The first checkpoint (groups share their ideas with the instructor) would work better after part three. Also, it is not clear in the lesson plan how this should be structured. I would prefer groups report directly to the instructor(s), but it may be more efficient to have a brief class discussion. Also, I forgot to have students think about measuring symmetry on their own. Actually, by this point in the activity most students had already discussed this. It may be better to incorporate part four of the activity into part one. I also ended up not getting groups started on their posters until they had come to consensus on both ranking (part three) and measurement (part five) of symmetries.

The activity ended up flowing this way:

1. Private consideration of rankings.
2. Discussing possible rankings and reasons for them.
3. Discussing how to measure symmetry, possibly affecting ranking decisions.
4. Coming to consensus on a ranking and measurement system.
5. Making a poster.

Neither class made it to part seven of the activity—not quite every poster was finished in time. This was a shame especially in first period, as different groups had decided on different measurement systems and rankings.

I plan to use this activity again in math circle, restructured to reflect the natural flow noted above and to require more group interdependence.

References

Sean Larsen, personal correspondence.
1. (private think time) Rank these seven figures in order from least to most symmetric. There is not one correct way to do this. Use your instinct, intuition, and artistic sensibility.

A.  B.  C.  D.  

E.  F.  G.  

2. (go-around) Share your ranking with your group.

**Checkpoint**

3. (group brainstorm) Come to consensus on one ranking of the seven figures. Prepare a poster that ranks the seven figures from least to most symmetric.

4. (private think time) Think about how you would measure (that is, assign a number to) the symmetry of a figure in a way that is consistent with your group’s ranking.

5. (group brainstorm) Come to consensus on a method for measuring symmetry, that is, a way to assign a number any figure that tells how symmetric the figure is.

**Checkpoint**

6. Create a poster describing your group’s method of measuring symmetry. Use the seven given figures to illustrate your method.

7. Poster Q & A and gallery walk. Visit the other groups and compare their ideas to those of your group.