LESSON PLAN

PURPOSE

- An introduction to the Golden Ratio and its geometrical appearance in a regular pentagon. Students will use facts they have learned throughout the year to discover (‘prove’) that the length of any diagonal inside a regular pentagon is in fact the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2} = 1.61803399...$. The goal is for students to see the effectiveness in using techniques of geometry and algebra to answer a mathematical problem. This goal is also to present an unexpected occurrence of the Golden Ratio in pure mathematics, in contrast to its occurrence in nature. This problem will be approached first by geometrical observations and second by algebraic manipulation. This lesson plan is suggested for the latter part of a geometry course or, for any students who have a good understanding of the necessary techniques.

GUIDING QUESTIONS

- What is the Golden Ratio, $\phi$?
- What is the length of any diagonal found in a regular pentagon with sides of length 1?

BRIDGE

Student need to know the following:

1. The interior angles of a four-sided object add up to 360°.
2. Euclid’s fifth postulate.
3. Recognize geometrical shape of a rhombus.
4. Recognize two triangles are similar by angle/angle.
5. Set up and solve algebraic equations using the quadratic formula.

STANDARDS

- This lesson addresses the geometry standards of recognizing geometrical objects, parallel lines, and similar triangles. It also brings in algebra standards of setting up an equation and solving using the quadratic formula.

MATERIALS

Hands-on material:

- One ruler and a pencil per student.
- Poster paper and markers.

Written material

- A hand-out with a partially labeled pentagon, and guiding questions to solve the problem step by step, see attached.
Lesson Outline

Launch
(10-15 minutes) Start the class with an introduction to the Golden Ratio or, if you have already discoursed its unique geometrical property, at least have the students talk about what they remember and recall some interesting observations. An introduction with warm up questions about \( \phi \) is attached. Make sure to emphasise that the precise definition of the Golden Ratio is given in an abstract geometrical settings while it is also found in the geometry of our every day lives.

Bridge
(25-30 minutes) Have the students get into groups and begin with the hand-out. Take time to emphasise this is a multi-step proof of a geometrical fact and that they should take their time to see how each step’s conclusion leads them to the next observation, all leading up to our final solution. This is a good exercise in practicing organizational skills for thinking through a multi-step proof. The teacher should move about the class room and talk to the students to see how different groups are handeling the questions/steps. Assign each group a different step in the proof to work out on the poster with their markers and ruler. A possible group assignment could be the following: Group 1 - questions 3(a) and 3(b), Group 2 - questions 3(c) and 3(d), Group 3 - question 4, Group 4 - questions 5 and 6, Group 5 - questions 7 and 8, Group 6 - question 9.

Closure
(10 minutes) Once the class seems to have worked through the hand out and made their posters, have a class discussion calling forth the respective goups to show their posters while we talk through the proof. We can end with a class discussion on other aspects of the golden ratio.

Pocket Questions

- What would happen to the length of the diagonals if I had started with a regular pentagon with sides on length 2? or any other length?

- Can you find any other occurances of the Golden Ratio in geometry, and can you figure out how these observations may be proven? (Here are two, but there are many others: (1) From the following picture show that the ratio of the radius of the larger circle to the diameter of the smaller circles is \( \phi \). (2) How can \( \phi \) be used to construct a regular icosahedron?)

\[
\text{Diagram:}
\begin{array}{c}
\text{Picture of circles arranged in a pattern showing the Golden Ratio.}
\end{array}
\]
What is the length of the diagonal in a regular pentagon with side of length 1?

1. With your ruler draw three specific diagonals inside the blue pentagon above: One must be horizontal and the two other diagonals must intersect each other but not the first one. Label this point of intersection $F$. (We could label it anything, but let’s stay consistent with the rest of the hand out)

2. Do the diagonal lines seem parallel to any other lines? (Hint: what about sides of the pentagon?)

3. Is the observation you made in question 2 seem true for lines $EB$ and $DC$? Can you prove it?
   (a) Consider the object $EDCB$, what can be said about the sum of the interior angles?
   (b) What do you notice about $\angle EDC$ and $\angle BCD$? What do you notice about $\angle BED$ and $\angle EBC$?
   (c) Use these fact to show that $EB$ is parallel to $DC$.
   (d) Does this argument work for any diagonal and opposite side of the pentagon?
4. With what was just shown and the fact that every side of our pentagon is of length 1, what do you notice about the object $EDCF$? Does this figure have a name? What is the length of its sides? Specifically, what is the length of $EF$?

5. What is the relationship between $EC$, $EF$, and $FB$?

6. If $EC$ and $FB$ are unknown let’s assign them variables $x$ and $y$, respectively. Now what does the relationship in question 5 become?

7. Can anything be said about $\triangle EFC$ and $\triangle AFB$? What geometrical fact proves your observation?

8. Using what was found in question 7, set up an equation involving $x$, $y$, and 1. Using what you know from question 6, solve for $y$.

9. With your answer for $y$, find the value of $x$, our unknown length of a diagonal. Does this number look familiar?
Lesson Notes

• Intro: This is a lesson plan that makes a geometrical observation about $\phi$, therefore I believe it would be good to start with a geometrical definition of $\phi$. This will involve students finding out the exact value of $\phi$ themselves by solving an algebraic equation the teacher/class sets up. This starts them off with a good warm up problem and will parallel the worksheet rather well. After that present some relationships between $\phi$ and nature, showing some nice pictures and models.

• First let’s present the geometrical definition: Let $x > 1$ and consider the following picture:

Can the class see a big rectangle, a square, and a small rectangle? Suppose that $x$ (the longer side of the big rectangle) is a special number such that if I remove a square from the big rectangle, the small rectangle will be similar the to big one. Recall that similar objects have porportional sides. Can we set up a proporion? How about the following,

$$\frac{\text{short side}}{\text{long side}} = \frac{1}{x} = \frac{x-1}{1}.$$  

Using this algebraic expression solve for $x$. Students should find that this expression is equivalent to $x^2 - x - 1 = 0$ and by applying the quadratic formula find that the positive solution is $x = \frac{1+\sqrt{5}}{2}$, this is what we call the Golden Ratio, an irrational number represented with the symbol $\phi$. The negative solution is also of some interest since it is actually $1 - \phi$. The rectangle above is what we call the Golden rectangle.

• To lead into the relationship of $\phi$ in nature, discuss the fact that the process of sectioning off a square may be continued and present a picture as a visual aid. This can then lead to drawing the ‘Golden Spiral’ which nature mimicks with shells, flowers, and pine cones.

• These observations can give students a nice meaningful association with the number $\phi$. Emphasize to the class how impressive these observations really are. The fact that $\phi$ is a number formulated in pure mathematics which is displayed in an abstract geometrical setting, while the same number shows up in the geometry of natural forms - objects in our daily lives. The teacher should tell students that this ratio shows up in the architecture of our buildings (i.e., long side/short side) and in the proportions of our bodies (i.e., the length of shoulder to fingers/the length of elbow to fingers), at least it is approximated with these examples.
After this introduction, prepare students for discovering how φ shows up in other unexpected geometrical settings, for example, the regular pentagon or the regular icosahedron.

**Solutions to the worksheet**

1. This step may seem a little strange, but the proof does not require the use of these specific diagonals (we could use any three diagonals as long as two intersect and the third does not). I chose these specific diagonals because I believe they help to illuminate the proof; to help give a clear picture of what is being talked about at each step. We can at least see it as a starting point for students to follow directions in drawing specific diagonals, showing that they have some understanding of what objects we are investigating.

2. The diagonals are parallel to their opposite sides.

3. This observation does indeed seem true for \( \overline{EB} \) and \( \overline{DC} \). The proof is worked out in the following questions.

   (a) The object \( EDCB \) has four sides, therefore the interior angles must sum to 360°.

   (b) Notice that \( \angle EDC \) and \( \angle BCD \) are interior angles of our regular pentagon and therefore they are equal. Also since \( \angle BED \) and \( \angle EBC \) are formed by the same diagonal and adjacent sides of our regular pentagon, they too are equal.

   (c) Let \( \angle EDC = \angle BCD = a \) and \( \angle BED = \angle EBC = b \). Then the equation \( \angle EDC + \angle BCD + \angle BED + \angle EBC = 360° \) is equivalent to \( 2a + 2b = 360° \) and therefore \( a + b = 180° \). Now if we were to extend \( \overline{EB} \) through the point \( E \) and extend \( \overline{DC} \) through the point \( D \), then \( \overline{DE} \) would be a transversal crossing our extended lines. Now we just showed that \( a + b = 180° \) which is the same as \( \angle EDC + \angle BED = 180° \). Therefore since the same side interior angles of the transversal crossing our extended lines sum to 180° the extended lines must be parallel. Thus \( \overline{EB} \) is parallel to \( \overline{DC} \).

   (d) This argument was not specific to this diagonal and opposite side of the pentagon. We just as easily could have shown \( \overline{EC} \) was parallel to \( \overline{AB} \), or any other pair.

4. The object \( EDCF \) is a four-sided object with parallel opposite sides. Therefore their lengths are equal and must be equal to 1. Specifically the length of \( \overline{EF} = 1 \). This object is called a rhombus.

5. The relationship is \( \overline{EC} = \overline{EF} + \overline{FB} \).

6. \( x = 1 + y \), since we also know that \( \overline{EF} = 1 \).

7. \( \triangle EFC \) and \( \triangle AFB \) are similar triangles by angle/angle. The first angle being the pair of vertical angles at the point \( F \), the second angle is seen by considering \( \overline{AC} \) as a transversal across lines \( \overline{EC} \) and \( \overline{AB} \). Therefore \( \angle ECA = \angle BAC \). Similarly, I could have chosen \( \overline{EB} \) as my transversal and found the equality of the other angles.

8. Since the triangles were similar their sides were proportional and we can set up the following equation:
\[
\frac{1}{x} = \frac{y}{1} \Rightarrow \frac{1}{1+y} = \frac{y}{1} \Rightarrow 1 = y(1+y) \Rightarrow 0 = y^2 + y - 1.
\]

Notice that this last equation is very similar to the equation solved in the introduction warm-up problem. Using the quadratic formula solve for \( y \), leaving it in symbolic form, a decimal expression will also work, but it will only be an approximation to the irrational value of \( \phi \).

9. The last step is to then solve for \( x \) to find that our diagonal of a regular pentagon is indeed the Golden Ratio \( \phi = \frac{1+\sqrt{5}}{2} \).

**Follow-Up Lessons**

- As a follow-up for the next lesson, teachers could work in the relationships to nature with the Fibonacci numbers and their connection to \( \phi \). This could then lead back to the Golden rectangle which can lead to a construction of a regular icosahedron. Three mutually orthogonal Golden Rectangles can give such a construction by using the Pythagorean Theorem twice. The picture below is the starting point.