$(CM)^2$ worksheet on completing the square.

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1 Why do we care to complete the Square?

When calculating such things as the vertex, foci, directrix, and axis of symmetry in regards to conic sections we find the skill of completing the square very useful. Understanding how to derive the necessary formulae for ourselves is a very important skill. What follows is a step by step process to that will lead you through the derivation of the formula to find the vertex of any parabola.

First we need to understand that the word parabolic refers to the shape of an object and that in general when we speak of a parabola we are referring to the graph of a quadratic function in one variable. That is to say the function in one variable $f(x) = ax^2 + bx + c$ with a, b, c real numbers. The right hand side of of this equation you will recognize as a second order polynomial. Many times we use the terms; **degree two polynomial**, **quadratic equation**, and **parabola** interchangeably. These terms actually have different meanings, although subtle.

By now you are familiar with the quadratic formula: $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$. These two values of x tell us where our function value is zero. Many students can use this formula, however the ability to derive this will allow us to look more deeply at conic sections.

1.1 Dare to complete the square

$0 = ax^2 + bx + c$	Set the function equal to zero	$0 = 2x^2 + 4x - 1$
	Divide through by the leading coefficient	$0 = x^2 + \frac{4}{2}x - \frac{1}{2}$
	Isolate the variable x on one side	$+\frac{1}{2} = x^2 + 2x$
	We can add any number to both sides of the equation	$\frac{1}{2} + \alpha = x^2 + 2x + \alpha$
	Imagine the right hand side as a perfect square	$\frac{1}{2} = (x + \triangle)(x + \triangle)$
	$\triangle + \triangle$ must equal the coefficient of x	$\triangle + \triangle = 2$
	This determines the value for $\triangle$	$\triangle = \frac{1}{2}2 = 1$
	This also determines the value of $\alpha = \triangle \times \triangle$	$\alpha = (\frac{1}{2}2)^2 = 1$
	The equation stays balanced	$\frac{1}{2} + 1 = x^2 + 2x + 1$
	Collect terms and write as a square	$\frac{3}{2} = (x + 1)^2$
	Take the square root of both sides, (why is there $\pm$?)	$\pm\sqrt{\frac{3}{2}} = x + 1$
	Isolate x and you are done	$-1 \pm \sqrt{\frac{3}{2}} = x$
	Check your result!	

1.2 How does this help me find the vertex of a parabola?

You have no doubt seen the vertex form of a parabola: $y = a(x-h)^2+k$. Given a quadratic function in standard from $f(x) = ax^2 + bx + c$ how is it that we can convert this to the vertex form? We can do this by taking note of the fact that we can manipulate an equation by operating on just one side.

Example 1.1. $y = \frac{1}{2}x + 4 = \frac{1}{2}(x) + 4 = \frac{1}{2}(x) + \frac{1}{2}\frac{2}{1}(4) = \frac{1}{2}x + \frac{1}{2}(8) = \frac{1}{2}(x + 8)$
This technique is known as force factoring.

Example 1.2. $y = 2x + 7 = 2x + 7 + 1 - 1 = 2x + 8 - 1 = 2(x + 4) - 1$
This technique is known as addition by zero, since we added then subtracted the same value.

Given $y = ax^2 + bx + c$ how do we find the values for h and k so that we can write this in vertex form?

To get us started lets regroup and use force factoring:

$$\begin{aligned} y &= ax^2 + bx + c \\ &= (ax^2 + bx) + c \\ &= a(x^2 + \frac{b}{a}x) + c \end{aligned}$$

Next we would like to form a perfect square in the variable x. This means there is some value we will need to add inside the parentheses that will create a perfect square.

$$y = a(x^2 + \frac{b}{a}x + \triangle) + c$$

Careful! If Triangle is any value other than zero our equation is out of balance and therefor false. What value needs to be added or subtracted back to the equation, but outside the parentheses? To discover this value follow similar steps to what you did above when completing the square.

Once you have a perfect square in the variable x you will have an equation that looks something like $y = a(x+m)^2 + n$. But wait! why did we not write this using h and k? Notice that the from we want is $y = a(x-h)^2 + k$. So that we may need to adjust the signs of m and n, but once thats done so are we.