Supercoiling: Linking Number, Twist, and Writhe

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1 Introduction & Background
DNA (deoxyribonucleic acid) is one of the most fundamental building blocks of life, and so it is an exciting fact that we can use mathematics to study the topological and geometric aspects of the famed double helix.

Figure 1: Each of the two phosphate backbones of double-stranded DNA can be represented by mathematical curves [1].
Figure 2: Here on the left, *Linking Number* = 2. This is found by counting the number of crossings (4) and dividing by two, as shown in the figure on the right.

For a circular piece of double-stranded DNA we can create a simplified model by idealizing the strands as two closed curves in space, as in Figure 1. To study these curves, mathematicians have created three quantities that describe the shape of the curves and their relationship to each other.

- The first is *linking number*, which is the number of times the two curves are intertwined. If you were able to pull the two curve apart as in Figure 2 then the linking number is obtained by counting the number of crossings and dividing by two. It is a topological invariant, meaning that amount of stretching or pulling will not change the linking number without cutting and reconnecting the strands.

- The second quantity is *twist*, which is most easily imagined as the number of times each of the curves rotates around the central axis $C$ of the double helix. This should not be confused with linking number, as the two-dimensional projection in Figure 3 shows; it is indeed possible to have a linking number of 2 but a twist of 0.

- The last quantity is *writhe*, which loosely put is the number of times the central axis $C$ makes loops about itself. Figure 4 shows the central axis (shown as a single, black line) taken from Figure 3 and has a writhe of 2.

A very interesting equation referred to as White’s Formula sums up the relationship between these quantities quite simply:

$$\text{Linking Number} = \text{Twist} + \text{Writhe}.$$ 

*Linking Number, Twist, and Writhe as Integers.*

Can any of the values in the above formula be negative? Of course they can! If we assign an orientation to each of the curves in Figures 2-4 (imagine each line is a road you are walking on, and there are arrows pointing in the direction you are walking), each of the crossings will be a rotation of one of the pictures...
Figure 3: Linking Number = 2; Twist = 0; Writhe = 2. To obtain the linking number, only count the four red-blue crossings (ignoring red-red and blue-blue). The twist is zero because if we imagine the red and blue paths to be the edges of a single, clear ribbon, then we see that the ribbon never twists and only has the same side facing up the whole time. The writhe is two because this ribbon crosses itself twice, as shown in Figure 4.

Figure 4: Writhe = 2. The “central axis” of the “ribbon” from Figure 4.
from Figure 5. Thus in Figure 4 the writhe as a signed quantity is actually -2! While this is not a focus of the lesson, it will be important for the teacher to be familiar with this convention as the lesson does allow students to explore these values as integers quite thoroughly.

What is a Projection?

In the above examples, we calculated the twist and writhe by representing the knots as a two-dimensional drawing, in essence taking a three-dimensional object and taking a photo in order to flatten it. The two-dimensional image is called a projection, and just as a three-dimensional object will look different from different angles, the twist and writhe values obtained from different projections can yield different values. Mathematicians have thus developed precise mathematical definitions for linking number, twist, and writhe that average these values over all possible projections. For more information, please see [2].

References


2 Learning Objectives

- Recognize the basic phenomenon of super-coiling
- Learn how to calculate writhe of a rope
- Learn how to represent writhe in a projection drawing
- Learn how to calculate twist of a rope
• Learn the relationship between writhe and twist
• Learn how mathematical concepts can be applied to the sciences (e.g., biology)
• Learn how to be methodical and organized when exploring new concepts

3 Materials Required

• Per student group (2-3 students) you need:
  – 1 small posterboard (sizes 8.5x11 to 8.5x14 are fine).
  – 2 binder clips.
  – A colored marker or pencil
  – 1 handout packet
  – 1 piece of duct tape
  – 1 twenty-inch piece of rope with a straight line drawn along the edge.

  * The rope should be of a solid color (except for the line) and it is preferred that braided nylon rope is used (not twisted). You can buy a longer rope, cut it into smaller pieces, and melt the loose ends so they do not fray. When the ends are still hot it is recommended to clamp them flat with pliers, so that when the rope is attached to the posterboard using the binder clips as in Figure 6, the melted end is held flat between the posterboard and the binder clip.

4 Lesson

The following lesson plans are for three 60-minute classes.

Day I (First Handout)

The first day of class is really an exploratory day. The students are introduced to the concept of supercoiling using a manipulative, and are asked to record in a table systematically their results. For the reader, take a minute to browse through “Day I” of the handout and then read the suggested classroom lesson plan below:

• Setup (5 min)
  – Partner up in groups.
  – Hand out the ropes, binder clips, colored pencils/markers, and packets to the groups.
• Introduction (10 min)
  – Ask for a volunteer to hold one end of a rope while you hold the other.
  – Demonstrate to the classroom what happens to the rope when one of you twists the end multiple times.
  – Introduce/discuss the concept of supercoiling.
    * Show a picture of DNA if you have the opportunity and mention that DNA like any rope also supercoils.
  – As a class fill out “Part I: Question”

• Have the students finish the packet (20 min)
  – Students that finish early can write their results on the board

• Classroom discussion (10 mins)
  – What do the students notice?
  – The discussion doesn’t need to come to any conclusions; however, more likely than not some students will notice that when the rope is twisted $n$ times, most probably $n$ loops will be formed.

• Clean-up (10 min)
Day II (Second Handout)
The second day develops the students’ comprehension of writhe as well as shows them the relationship between writhe and the twisting of the rope (twist). Again, peruse the handout for the second day and then read below:

- Setup (5 min)
  - Partner up in groups.
  - Hand out the ropes, binder clips, colored pencils/markers, packets, and tape to the groups.

- Warm-Up (10 min)
  - Have students as groups work on the warm-up.
  - Make sure you check in with every group.
  - Review as a class

- Have the students finish the packet (25 min)
  - Make sure that in “Part III: Writhe and Twist” the students label the writhe crossings with “+1”.

- Class Discussion (10 min)
  - At this point the students will see that every time you get rid of a writhe loop, you get an extra twist in the rope. This is the most important observation. Aim to as a class arrive at the formula

\[
\text{Writhe} + \text{Twist} = 3
\]

or even

\[
\text{Writhe} + \text{Twist} = \text{Constant}.
\]

  - If you have time, talk about the term Linking Number as the name given to the number of times you twist the rope before you clip it down. If you so wish, you can instead ask the class to come up with a name for this quantity.

- Clean-up (5 min)

Day III (Third Handout)
The third day is an exploratory day which lets students do a number of things: whether it’s basic counting, pattern recognition, recognizing symmetry, or adding/subtracting integers. Chances are different groups will benefit differently, allowing the discussion time to be even more fruitful. Again, read the handout and then follow below:
• Setup (5 min)
  – Partner up in groups.
  – Hand out the ropes, binder clips, colored pencils/markers, and packets.
• Introduction (10 min)
  – Do an example for question one as a class, labelling the crossing with “+1”, and writing Writhe = +1.
    * Make sure to say there might be more than one solution.
  – Take the opportunity to talk about positive and negative writhe and draw the images from Day II/Part II on the board to remind the class.
  – Do an example for question number two, making sure to label the crossings.
  – Ask the class for another example for question number two.
    * If they have trouble, you can ask them to refer to the handout from Day II (Part II, Question 2).
• Have the students complete the packet (20 min)
  – Again, make sure that the students label the crossings with “+1” or “-1” and write the total writhe for each diagram.
  – Many groups will only have positive crossings. As you are circulating you can ask these groups:
    * Ask them if they have found all the solutions with only positive crossings.
    * If they have studied symmetry or reflections, you can ask them to analyze their solutions in those terms.
    * Ask them to prepare a presentation for the class
  – Other groups might have a mixture of positive and negative crossings. As you are circulating you should,
    * Ask them to show you how they calculated the writhe (adding +1’s and -1’s)
    * Ask them to prepare a presentation for the class
  – Students that finish early can write their results on the board
• Class Discussion (15 min)
  – Presentations from the groups that you asked to prepare a short presentation
  – If you have time, give students time to reflect on the activities of the three days.
• Clean-up (5 min)
5 Reflection

Originally this lesson was designed for just one 90-minute class period, but it became obvious the first time I presented it that it would be better to spread it out over several days. I presented the first two days of the lesson to a Pre-Calculus high-school class at June Jordan School for Equity and all three days to a group of 4-5th graders in the Math Circle enrichment program at San Francisco State University. In all, most of the students were engaged by the activity but surprisingly the high-school students were less so. I attribute this to the fact that the material in this lesson would not be considered “mathematics” by many high school students, thus rendering it quite inconsequential in the grand scheme of their educational lives. The elementary school students on the other hand just saw it as a fun puzzle. I would like to point out to the reader that my experience is only anecdotal and to take it with a grain of salt. Your milage may vary.

It is not imperative that all of the students be able to recognize negative writhe vs. positive writhe. In fact, the lesson is designed so that some students will pick up on it, and the ones that don’t won’t be encumbered in any real way. I designed the lessons in this manner after my experience during the second day of the high school class when I drew the arrow diagrams in Figure 5 on the board. Big mistake. Although the diagrams are quite standard, they are cumbersome to work with and actually don’t help the students develop any “supercoiling sense” at all. I would present the arrow diagrams instead in a fourth lesson, in which students would be able to take a projection drawing with many crossings, assign an orientation arbitrarily, and then figure out the sign of each crossing.

I was pleasantly surprised by the Elementary school students who remained...
engaged throughout all three days and in some cases took it a lot farther than I had originally planned. On the third day one student came up with a method to create an infinite number of answers for question 1 (see the handout for “Day III”). She observed not only that it was possible to have just one writhe crossing as shown in Figure 7, but also that by adding to that an equal number of negative and positive writhe crossings, she could form 3, 5, 7, or even 9 crossings and still only have a writhe of -1. The only thing that stopped her from reaching 11 was the fact that “the rope was too short!”
Supercoiling Day I: Introduction

Part I: Question

Hold a piece of rope between your hands and twist one end 10 times. What happens in the middle of the rope? What happens if you pull the ends out/push them in? Draw a picture.

Part II: Exploration

- Place the cord on the table so that the line is facing up. Turn the right side of the cord toward you 1 time. Secure both sides.

- Now, flatten the cord on the table so that the line is only on the top. Show the teacher.

Draw the picture showing the shape of the cord below:
Now fill out the table following the same process:

<table>
<thead>
<tr>
<th>Turn the cord:</th>
<th>Draw the result:</th>
</tr>
</thead>
<tbody>
<tr>
<td>RIGHT side AWAY from you 1 time</td>
<td></td>
</tr>
<tr>
<td>RIGHT side AWAY from you 2 times</td>
<td></td>
</tr>
<tr>
<td>RIGHT side TOWARDS you 1 time</td>
<td></td>
</tr>
<tr>
<td>RIGHT side TOWARDS you 2 times</td>
<td></td>
</tr>
<tr>
<td>LEFT side AWAY from you 1 time</td>
<td></td>
</tr>
<tr>
<td>LEFT side AWAY from you 2 times</td>
<td></td>
</tr>
</tbody>
</table>
Supercoiling Day II: Writhe

Part I: Warm-Up

1. Copy the pictures of DNA on the left using your rope.
   - The black line should be on top.
   - When you are finished, connect the ends of the rope with tape.

![Form I](image1.png) ![Form II](image2.png)

2. Unravel the rope to get the figure on the right. How many times does the black line **twist** around the rope?
Part II: Exploring Writhe

1. Make the string look like the picture. Here we say the **writhe** is 0:

   ![String](image)

   Make the string look like this, and ask the teacher to check it. The **writhe** is **+1**:  
   ![String](image)

   Now, make the string look like this. The **writhe** is **-1**:  
   ![String](image)

   The difference?
2. In both of these the **writhe** is $+2$. Copy them with your string.

3. Make the string look like the following diagrams. What is the writhe of each one? (Hint: count the crossings)
Part III: Writhe and Twist
Make the string look like this picture with the line on the top and then clip both sides.

a. Here you have a writhe of $+3$ and the black line twists around the rope 0 times.

b. Undo one of the loops. We have a writhe of ___ and the black line twists around the rope ___ times. Draw it:

c. Undo one of the loops. We have a writhe of ___ and the black line twists around the rope ___ times. Draw it:

d. Undo one of the loops. We have a writhe of ___ and the black line twists around the rope ___ times. Draw it:

DISCUSS: Is there a pattern for the writhe and the twist?
Supercoiling Day III: Exploration

1. Lay the rope flat with the line on the top and twist the right end **ONE** time **TOWARDS** you. Draw all of the ways you can arrange the rope so that the black line is only on the top.

2. This time, twist the right end **TWO** times **TOWARDS** you. Draw all of the ways you can arrange the rope so that the black line is only on the top.
3. This time, twist the right end **THREE** times **TOWARDS** you. Draw all of the ways you can arrange the rope so that the black line is only on the top.
4. Lay the rope flat with the line on the top and twist the right end **FOUR** times **TOWARDS** you. Draw all of the ways you can arrange the rope so that the black line is only on the top.