1 Introduction & Background

As an introduction to dynamical systems, the author’s advisor posed the following problem, to which the results were previously known:

Given a set of vertices \( p_1, p_2, p_3, p_4, \) and \( p_5, \) connect them in order so that the edges of the polygon are \( p_1p_2, p_2p_3, p_3p_4, p_4p_5, \) and \( p_5p_1. \) [This polygon is not necessarily convex.] Draw the midpoint of each edge. Let the midpoint of \( p_ip_{i+1} \) be \( p'_i. \) Then the edges of the new polygon will be \( p'_1p'_2, p'_2p'_3, p'_3p'_4, p'_4p'_5, \) and \( p'_5p'_1. \) What is the behavior of the polygon as this process is iterated to infinity?

To answer this question requires a fair amount of linear algebra, which is not expected of the participants in the proposed activity to follow. However, it is beneficial for the instructor to be aware that the action of connecting midpoints can be expressed as a linear operator and that this linear operator can be further expressed in terms of its eigenbasis. By understanding the properties of the transformation through the linear operator it is clear that with probability 1, the polygon will become a regular pentagon. This is due to the fact that the eigenvectors which have coefficients with the largest modulus are the regular pentagons, thus as the iterative process continues to infinity, these regular pentagons are the last to diminish.

While a rigorous mathematical explanation of this behavior is above the ability of a high school or middle school student, observing this phenomenon happens after only a few iterations which makes this problem especially approachable. The aim of this activity is not to discuss linear algebra (except in some rare cases) but rather to allow students to practice geometry constructions, make observations about their work, discuss observations and make conjectures with their group, and present and defend their claims. While the rigor of math is noticeably absent, the process of doing mathematics is at the center of this activity.

2 Learning Objectives

(I) Orienteering

   (i) Measuring angles
   (ii) Determining smallest/largest angles
   (iii) Finding midpoints
   (iv) Measuring line segments
(II) Critical Thinking

(i) Inferring long term behavior
(ii) Stating conjectures
(iii) Presenting conjectures
(iv) Challenging/Defending conjectures of others

3 Materials

(I) Rulers (1 per student)
(II) Colored Pencils (1 pack per group of 4)
(III) Protractor (1 per student)
(IV) Graph Paper (1 per student)
(V) Chart Paper/Poster Paper (1 per group of 4)
(VI) Colored Markers (1 pack per group of 4)

4 Instructional Plan

This activity requires a brief introduction for the method and is left to the student for execution and observation.

(I) Teacher-led Example:

Using a document camera or the board draw five dots as far apart as permissible. Label the points a through e or 1 through 5. Connect the points in order, i.e. there should be only segments $ab, be, cd, de$, and $ea$. Determine the smallest and largest interior angles of this polygon. Record them to the side.

Next, mark the midpoint of each line segment using a ruler. Be as precise as possible. Maintaining the order used in the first step, connect the midpoints together using a different colored marker. Record the smallest and largest angles again.

From here the students should understand the process and be able to iterate it several more times while recording the data on the worksheet provided. [Section 6]
(II) Student Work:

Students will be seated at tables with four positions: Facilitator, Resource Manager, Recorder/Reporter, and Team Captain.

(A) Facilitator: The facilitator’s responsibility is that of starting the group by reading the directions aloud, or designating someone from the team read the directions aloud. The purpose of this task is to ensure that each team member understands the activity.

(B) Resource Manager: The resource manager’s responsibility is to collect all materials necessary for the activity. This helps maintain classroom control by only allowing a small portion of students to roam for supplies. Secondly, the resource manager is also responsible for gaining the instructors attention for team questions.

A team question is a question posed by one team member which cannot be answered by any of the other team members.

(C) Recorder/Reporter: The recorder/reporter’s responsibility is to neatly and methodically record any information that may need to be shared upon the completion of part of all of the activity. The recorder/reporter is also responsible for reporting findings from the activity or to designate the reporting of such findings.

(D) Team Captain: The team captain’s responsibility is to ensure that the group is working as a whole, enforce classroom norms, and to fill any position left open.

Students will begin by recreating the example shown previously by the instructor. Each student will plot their own points, hopefully with some significant variation, though this is not to be mentioned by the instructor. When the students have successfully drawn the second polygon using the midpoints, a checkpoint is required before moving forward.

A checkpoint is when the team stops having completed a designated portion of the activity to be reviewed by the instructor before moving forward. A checkpoint ensures that teams are working together and are understanding the concepts.

Once the students have drawn 3 iterations and completed the table they will be asked to record their observations.
(III) Student Observations

(A) The shape becomes smaller.

(B) The difference between the smallest and largest angles becomes smaller.

(C) The shape becomes more regular.

(D) The shape becomes convex/stay convex.

(E) The shape rotates.

(IV) Instructor Led Discussion

Upon completion of the worksheet, or as dictated by time, the instructor leads a discussion covering the student observations. The recorder/reporter of each group (or a member designated by the recorder/reporter) will report one finding different from those previously stated by other groups. When a team states their observation they should also provide an explanation of why it’s true. This may range anywhere from intuition to more complex geometry.

If any groups were able to find invariant polygons (Section 5), they should come to board and draw them.

(V) Gallery Walk (Alternative)

Students may cut and paste their polygons onto poster paper and state the observations there. Once hung, other teams may view the poster.

5 Extensions

Here we discuss some extension questions for the observations that are made by students.

(A) The shape becomes smaller.
   At first this seems like an obvious and uninteresting observation and the instructor should expect that many students state this. To extend this the instructor can ask what the area will be after infinitely many iterations. This leads to a discussion on the idea of infinity and limits.

(B) The difference between the smallest and largest angles becomes smaller.
(C) The shape becomes more regular.

(D) The shape becomes convex/stay convex.
Items (B), (C), and (D) are the same action. This occurs because the action of connecting midpoints is a linear transformation. When a five sided polygon is listed as a linear combination of its eigenvectors the leading coefficient with the largest modulus is the pentagon, thus over many iterations, the shape is trending toward a pentagon. While interesting, this conversation might be saved for the most advanced students.

(E) The shape rotates.
By virtue of using midpoints, the vertices will never be rotated to a pre-existing vertex. An extension might involve choosing one vertex and noting the way it moves around the origin, using the positive $x$-axis as $0^\circ$. Over time, the rotation will become regular. The instructor may be able to find the limit of the angle of rotation. This again, relates to the shape becoming regular over many iterations.

Another extension is to ask the students if they can find any polygons that do not change shape under the action of connecting midpoints. Students will likely guess that the pentagon is such a shape. More interesting, is that the pentagram also has this characteristic. All the invariant polygons are precisely the eigenvectors of the transformation. Moreover, each shape is two shapes since we are considering orientation (i.e. there is a pentagram for both clockwise and counterclockwise motions).
6 Worksheet

Geometry
Fall 2012

Down The Rabbit Hole

<table>
<thead>
<tr>
<th>Facilitator</th>
<th>Resource Manager</th>
</tr>
</thead>
<tbody>
<tr>
<td>Get the team off to a quick start by making sure someone reads the task card. Make sure everyone understands the task. Check in with team members to keep the group moving.</td>
<td>Collect supplies the team needs to complete the task and organize the clean-up. Call the teacher over for team questions and check-points.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Recorder/Reporter</th>
<th>Team Captain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make sure everyone’s ideas get recorded. Make sure each person on the team is ready to explain your team’s thinking.</td>
<td>Enforce the norms for teamwork and help your team find compromises if needed.</td>
</tr>
</tbody>
</table>

(1) Each person should plot five points on his or her graph paper. Make them far apart.

(2) Label the points.

(3) Connect the points *in order*.

(4) Measure the smallest and largest angles. Record the results in your table on the next page.

(5) Find the midpoint of each line segment. Use a ruler.
(6) With a different colored pencil, connect the midpoints in order.

(7) CHECKPOINT! ________

(8) Measure the smallest and largest angles. Record the results in your table.

(9) Repeat two more times.

<table>
<thead>
<tr>
<th>Figure Number</th>
<th>Smallest Angle</th>
<th>Largest Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
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<td>2</td>
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<tr>
<td>3</td>
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</tbody>
</table>

What do you notice? Name at least two different things

(a) 

(b) 

As a group, form a conjecture about what will happen if this process continues. Record your conjecture on chart paper. Be ready to share your results with the class.
7 Assessment Options

The instructor may choose to assess the groups in the following ways.

(I) Participation Quiz:
The instructor may observe groups working together and assign value to the manner in which teams adhere to the classroom and group norms. Typically points are given when a team question is asked, when the team waits for all members to be finished before moving forward, when a team member helps another team member, when the team fulfills the specific roles (facilitator, recorder/reporter, resource manager, team captain), and when the team is staying on task.

(II) Group Quiz:
The instructor may choose to assign this as a group quiz and grade the worksheet for thoughtfulness, thoroughness, and completion.

(III) General Classwork:
The instructor may choose to grade this solely as they would a typical classwork assignment, looking for full or partial completion.

8 Instructor’s Reflection

The instructor may find that students vary dramatically in their observations. Being prepared with leading questions while circulating the room will allow students to explore an idea, even if it is not their own realization. Another helpful suggestion for challenging groups would be asking them if they could create a polygon that does not change shape, but only size. This activity can be a nice segue into the idea of a limit and may be appropriate for a relaxed day in pre-calculus. Introducing the idea of infinity can be very intuitive for students and this activity certainly lends itself to such discussions. Overall, students preformed well and had insightful commentary regarding their work and the work of others.