(CM)$^2$: Outline for introducing sequences and series.

Nick Dowdall
San Francisco State University
May 18, 2011

1 Introduction

Consider the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots, n, \ldots\}$ and the set of all non negative rational numbers $\mathbb{Q}^* = \{\frac{a}{b}, a, b \in \mathbb{N}, b \neq 0\}$.

What differences do you see between these two number sets?

In the set $\mathbb{N}$ what numbers, if any, are between 3 and 4?

In the set $\mathbb{Q}^*$ what numbers are between 3 and 4?

How many numbers are between 3 and 4? How many numbers are between $\frac{1}{2}$ and $\frac{2}{3}$?

Now what do you think the main difference is between $\mathbb{N}$ and $\mathbb{Q}^*$? This property is called being dense in itself.

Now consider the two lists of numbers:

$L_1 = \{0, 2, 4, 6, 8, \ldots\}$

$L_2 = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots\}$

The commas tell us there is nothing between two consecutive terms in the list and the ellipsis tells us that the lists continue on in exactly the way you would expect. Compare these two lists to $\mathbb{N}$
and $\mathbb{Q}^*$. Which one are they similar to?

We could actually think of these lists as being a function. The input values come from $\mathbb{N}$. There is a 0\textsuperscript{th} term, a 1\textsuperscript{st} term, a 2\textsuperscript{nd} term and so on in each of the lists $L_1$ and $L_2$. The terms themselves are different but they can be ordered in the same way that $\mathbb{N}$ is ordered. So it is very natural to use $\mathbb{N}$ as our input values. $\mathbb{N}$ is often referred to as the index set.

For this reason a sequence is defined as a function whose domain is $\mathbb{N}$ and the range is the list of numbers we see. Do $L_1$ and $L_2$ fit the definition of a sequence?

What is their range?

## 2 Arithmetic Sequences

Consider the sequence $L_1 = \{0, 2, 4, 6, 8, \ldots\}$. If you pick any two consecutive terms what is the difference between them?

What about $L_2 = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots\}$?

If a sequence has a common difference between any two consecutive terms like $L_1$ and $L_2$ we say that these sequences are arithmetic. We can write out the first 5 terms of $L_1$ using only the first term and some multiple of the common difference. \{0 + (0 \cdot 2), \ 0 + (1 \cdot 2), \ 0 + (2 \cdot 2), \ldots\}. This implies a formula for the $n$\textsuperscript{th} term in the sequence. If we let $a_n$ be the $n$\textsuperscript{th} term of our sequence then

\[
\begin{align*}
   a_n & = (\text{starting value}) + \text{common difference} \cdot (n - 1) \\
   & = a_1 + d(n - 1) \\
   & = 0 + 2(n - 1)
\end{align*}
\]

This form of writing the $n$\textsuperscript{th} term of the sequence does not rely on knowing the $(n - 1)$\textsuperscript{th} term. It
is often referred to as the closed form of the sequence. Can you find the closed form for \(L_2\)?

There are other lists of numbers that fit the definition of sequences but are not arithmetic. Consider

\[L_3 = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}\]
\[L_4 = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}\]

Do these two lists fit the definition of sequence? Is there a common difference between consecutive terms? Can you explain in words what these sequences are?

### 3 Geometric Sequences

By now you may have guessed that there are many different types of sequences. The two main types we wish to study are arithmetic and geometric. A geometric sequence is one in which there is a common ratio between consecutive terms. If \(a_i, a_{i+1}\) are two consecutive terms in a geometric sequence, by convention we say \(\frac{a_{i+1}}{a_i}\) is this common ratio, \(r\). Are \(L_3\) and \(L_4\) geometric? If so what is the common ratio?

Similarly to the last section, we would like to find a closed form for a geometric sequence. That is to say, we would like to find a simple rule for the \(n^{th}\) term. The key to this is in viewing each term of the sequence as some multiple of the first term \(a_1\). Write out the first few terms of \(L_3\) using only the common ratio \(\frac{1}{2}\) and the first term 1. What would the \(n^{th}\) term look like?

Thus for any geometric sequence we can write \(\{a_1r^0, a_2r^1, a_3r^2, \ldots, a_nr^{n-1}, \ldots\}\) so that our closed form is \(a_n = a_1r^{n-1}\). Of course this makes no sense if \(r = 0\).

Even though \(L_4\) is not arithmetic nor geometric, can you write a simple closed form for the \(n^{th}\) term?
4 Serries

What if we want to sum up part or all of the terms in our sequences? This is a particularly interesting area of mathematics and has long fascinated mathematicians.

If I take a sequence and place a + between each term I call this a series. Consider the sequence $L_1 = \{0, 2, 4, 6, 8, \ldots\}$. I can convert this to a series by writing $S = \{0 + 2 + 4 + 6 + 8 + \ldots\}$. This would mean sum up all of the even positive integers. There is a short hand for this idea using the Greek letter Sigma, $\sum$. We also need a closed form for the $n^{th}$ term. We already found this was $0 + 2(n - 1)$. So we could write this as $\sum 2(n - 1)$. But we need one more thing. We need to know where to start and stop the summation. If we never stop we would just say $\infty$. So in the case of the series $S = \{0 + 2 + 4 + 6 + 8 + \ldots\}$, $S = \sum_{n=1}^{\infty} 2(n - 1)$. In this case, the sum is infinite.

In fact is it worth pondering for just a moment what happens if we sum the same number an infinite number of times? What must our result be? What is $\frac{1}{10}$ summed an infinite number of times?

If we want a finite answer we often can only consider a portion of our series. Consider $\sum_{n=1}^{10} 2(n - 1)$. This is asking, what is the sum of the first 10 non-negative integers. If the underlying sequence is arithmetic, we can discover a very nice formula for this sum. Write the series twice, but one in reverse order.

$$S_{10} = 0 + 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18$$
$$S_{10} = 18 + 16 + 14 + 12 + 10 + 8 + 6 + 4 + 2 + 0$$

Now what if we added these two rows together? We would get
\[ 2S_{10} = 18 + 18 + 18 + 18 + 18 + 18 + 18 + 18 + 18 + 18 \]
\[ = 10(18) \]

But this sum is twice as large as we wanted. Thus \( S_{10} = \frac{10(18)}{2} \). Let’s try that again. Let \( S = 1 + 2 + 3 + 4 + \ldots \). Using the same method, can you sum the first 100 terms? That is to say, what is \( S_{100} \)?

This method is known as a Gaussian sum, or sometimes ”little Gauss” named after a famous mathematician by the same name. Notice that the we only need to know the first term and the last term of the portion of the series we are interested in to discover the sum. That is, we add \( a_1 + a_n \) which tells us what each term looks like in our double sum.

\[
S_n = a_1 + a_2 + a_3 + \ldots + a_n
\]
\[
S_n = a_n + a_{n-1} + a_{n-2} + \ldots + a_1 \text{ but this series is arithmetic so that}
\]
\[
S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d)
\]
\[
S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_n - (n-1)d) \text{ so that}
\]
\[
2S_n = n(a_1 + a_n)
\]
\[
S_n = \frac{1}{2}(a_1 + a_n)
\]

So we can see how useful it is to have the closed form for \( a_n \). Is it possible to ever have a finite value for and infinite arithmetic series? Why or why not?
What type of infinite series can we sum up? We know that the terms must get smaller and smaller or else we would be in the situation of summing and infinite amount of the same number again, or worse! Thus, the individual terms must approach zero. As it turns out not all series with this property have a finite sum. \[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots
\] does not have a finite sum even though the individual terms get smaller and eventually go to zero. This series is known as the harmonic series and a student usually proves its divergence in a second course in calculus.

There are well behaved infinite series that do have finite sums and in fact they have a very nice formula. Recall the geometric sequence we discussed earlier \( L = \{1, \frac{1}{2}, \frac{1}{4}, \ldots\ \} \) which had the closed form \( a_n = 1 \cdot (\frac{1}{2})^{n-1} \). The series would then look like \( \sum_{n=1}^{\infty} 1(\frac{1}{2})^{n-1} \). Write out the first few terms to convince yourself this is correct. Is it possible this infinite series has a finite value. This is a very famous series that goes back to ancient Greece and was know as Zeno’s paradox. Well to be exact, we would need to remove the first term of the series to make it into Zeno’s paradox. I leave it to the reader to look up this fascinating riddle that dates back to B.C. As it turns out the sum we just listed has a finite value, 2. This should be rather shocking. How can we prove this and more importantly, how can we find other sums of similar series?

As before, write out the terms of the series but then multiply the series by the common ratio \( r = \frac{1}{2} \) and write it out again:

\[
S = 1 + 1 \left( \frac{1}{2} \right)^1 + 1 \left( \frac{1}{2} \right)^2 + 1 \left( \frac{1}{2} \right)^3 + \ldots
\]

\[
\frac{1}{2} S = \left( \frac{1}{2} \right)^1 + \left( \frac{1}{2} \right)^2 + \left( \frac{1}{2} \right)^3 + \ldots
\]

Now subtract the bottom line from the top. What cancels? Once you reorder things and solve for \( S \) you will find a very nice result. Next do this in general. Just remember that every term of a geometric series is just \( a_1 \) multiplied by some power of the common ratio \( r \). So every term can be written as \( a_1 r^k \) for some non-negative integer \( k \). Follow what we just did and you will find a very
nice formula for any geometric series. One thing should be noted. What if you only want a finite number of terms summed? The formula for this can be found in exactly the same way as for the infinite case, but now there is one extra term that won’t cancel. Which is it?
Thus for arithmetic and geometric series you have found very nice formulae that will help you quickly take sums.