Complex Numbers on $\pi$ day

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1 Introduction

For Advanced Algebra classes who have been introduced to complex numbers, this activity is a great way to bring higher mathematics into the classroom. Complex Analysis is a whole field of math that high school students would not normally encounter. Although this activity could be done on any day of the year $\pi$ Day is already a day for fun and math. Using complex numbers and $\pi$ we can derive a surprisingly simple identity. The lesson is about proving that $e^{i\pi} + 1 = 0$.

2 Mathematical Background

$\pi$

$\pi$ is a Greek letter, pronounced “pi”. It is used to represent the ratio of the circumference a circle and its diameter. It’s approximate decimal representation is

$$\pi = 3.14159...$$

$\pi$ is an irrational number which means that the decimal expansion never repeats or ends. A chronology of $\pi$ can be found in [1]. A video illustrating the relationship between $\pi$ and the circumference of the unit circle can be found in [2].

$i$

Imaginary numbers were used in the 16th century to solve cubic and quartic polynomial, polynomials of degree three and four, respectively. The imaginary number $i$ is defined to by $i^2 = -1$. Complex numbers are of the form

$$a + bi$$

where $a, b$ are real numbers and $i = \sqrt{-1}$. 
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\[ e^{i\pi} + 1 = 0 \quad \pi = 3.14159... \]

$e$

The number $e$ is a mathematical constant. It’s approximate decimal representation is

\[ e = 2.71828... \]

e is an irrational number like $\pi$. $e$ is used frequently in calculus and is defined by the limit of $(1 + \frac{1}{n})^n$ as $n$ gets larger and larger. More information and history about $e$ can be found in [4].

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Euler’s Formula

The unit circle is a circle of radius one centered at the origin. There are $360^\circ$ (degrees) or $2\pi$ radians in a circle. Given the a point $(x, y)$ on the unit circle, pictured in Figure 1,

\[ a = \cos(\theta) \quad \text{and} \quad b = \sin(\theta) \]

Euler’s formula says that for any $\theta$,

\[ e^{i\theta} = \cos(\theta) + i \sin(\theta) \]

![Figure 1: Complex unit circle](image-url)
Complex Numbers on π day

\[ e^{i\pi} + 1 = 0 \]

\[ \pi = 3.14159... \]

3 Lesson Plan

This lesson plan can be done in 30 minutes. Students must be familiar with the trigonometric functions cosine and sine and the complex numbers.

1. March 14 is π Day.

\[ \pi = 3.14159... \]

If needed, a brief reminder of π can be given.

2. Remind the students about complex numbers, \( a + bi \) where \( i = \sqrt{-1} \).

3. If necessary, talk about \( e \).

4. Remind the students about the unit circle they learned about in Geometry class and the trigonometric functions cosine and sine. For a point \( a + bi \) on the unit circle

\[ a = \cos(\theta) \quad \text{and} \quad b = \sin(\theta) \]

5. Introduce the Euler’s formula from Complex Analysis that:

\[ e^{i\theta} = \cos(\theta) + i \sin(\theta) \]

Euler’s formula is an important theorem in Complex Analysis. More information can be found [3].

6. Ask the students if they can figure out how to prove that

\[ e^{i\pi} + 1 = 0 \]

Hint that the unit circle and Euler’s formula will be the key to proving this theorem. Give them 5-10 minutes to think about it.

7. Ask the students if they have come up with any ideas. If they have talk about it as a class.

8. If none of the students figured out a proof, show them how to prove it.

Proof of Euler’s formula:

\[ e^{i\pi} = \cos(\pi) + i \sin(\pi) \]

Since the \( \cos(\pi) = -1 \) and \( \sin(\pi) = 0 \) we find that,

\[ e^{i\pi} = (-1) + i(0) = -1 \]

Adding 1 to both sides we find the identity,

\[ e^{i\pi} + 1 = 0 \]
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\[ e^{i\pi} + 1 = 0 \]

\[ \pi = 3.14159... \]

4 References

1. [http://www-history.mcs.st-and.ac.uk/history/HistTopics/Pi_chronology.html](http://www-history.mcs.st-and.ac.uk/history/HistTopics/Pi_chronology.html)


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\[ e^{i\pi} + 1 = 0 \]

\[ \pi = 3.14159... \]

Solutions to worksheet:

1. \( \cos(\pi) = 1, \cos\left(\frac{\pi}{2}\right) = 0, \)
2. \( \sin(\pi) = 0, \sin\left(\frac{\pi}{2}\right) = 1, \)
3. \[ e^{i\pi} = \cos(\pi) + i\sin(\pi) \]
   Since the \( \cos(\pi) = -1 \) and \( \sin(\pi) = 0 \) we find that,
   \[ e^{i\pi} = (-1) + i(0) = -1 \]
   Adding 1 to both sides we find the identity,
   \[ e^{i\pi} = i \]
4. \[ e^{\frac{\pi}{2}i} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) \]
   Since the \( \cos\left(\frac{\pi}{2}\right) = 0 \) and \( \sin\left(\frac{\pi}{2}\right) = 1 \) we find that,
   \[ e^{\frac{\pi}{2}i} = (0) + i(1) \]
   \[ e^{\frac{\pi}{2}i} = i \]
π is the ratio of the circumference of a circle and it’s diameter. \( \pi = \frac{C}{d} \)
where \( C \) is the circumference of the circle and \( d \) is the diameter.

\[
\pi \approx 3.14159265358979323846264338327950288... 
\]

\[
e \approx 2.718281828459045235360287471352...
\]

**Euler’s formula:** for any \( \theta \),

\[
e^{i\theta} = \cos(\theta) + i\sin(\theta)
\]
where \( i = \sqrt{-1} \).

1) What is \( \cos(\pi) \)? \( \cos\left(\frac{\pi}{2}\right) \)?

2) What is \( \sin(\pi) \)? \( \sin\left(\frac{\pi}{2}\right) \)?

3) Using Euler’s formula and the unit circle prove that

\[
e^{i\pi} + 1 = 0
\]

4) Using Euler’s formula and the unit circle prove that

\[
e^{\frac{\pi}{2}i} = i
\]