(CM^2) Math Circle Lesson:
Game Theory of Gomuku and (m,n,k-games)

Overview:
Learning Objectives/Goals: to expose students to (m,n,k-games) and learn the general history of the games throughout Asian cultures.

SWBAT… play variations of m,n,k-games of varying degrees of difficulty and complexity as well as identify various strategies of play for each of the variations as identified by pattern recognition through experience.

Materials:
Paper and pencils for everyone

Vocabulary:
Game – we will create a working definition for this….
Objective – the goal or point of the game, how to win
Win – to do (achieve) what a certain game requires, beat an opponent
Diplomacy – working with other players in a game
Luck/Chance – using dice or cards or something else “random”
Strategy – techniques for winning a game

Agenda:
Check in (10-15min.)
Warm-up (10-15min.)
Lesson and game (30-45min)
Wrap-up and chill time (10min)

Lesson:

Warm up questions:
Ask these questions after warm up to the youth in small groups. They may discuss the answers in the groups and report back to you as the instructor. Write down the answers to these questions and compile a working definition. Try to lead the youth so that they do not name a specific game but keep in mind various games that they know and use specific attributes of them to make generalizations.

- What is a game?
- Are there different types of games?
- What make something a game and something else not a game?
- What is a board game?
- How is it different from other types of games?
• Do you always know what your opponent (other player) is doing during the game, can they be sneaky?
• Do all of games have the same qualities as the games definition that we just made? Why or why not?

**Game history:**
The earliest known board games are thought of to be either ‘Go’ from China (which we are about to learn a variation of), or Senet and Mehen from Egypt (a country in Africa) or Mancala. Which one of these games is the oldest is hard to say but it is known that Senet and Mehen boards have been found that date back as far as 6000 years ago. That makes them the oldest known board games to be found intact. While it is not clear how to play the games, since no one ever bothered to write down the rules to them, historians are sure they are games nonetheless. The next oldest known game is that of Ur, found in ancient Mesopotamia or what is now known as Iraq.

While these games are old, it is thought that there must have been much older games that weren’t played without boards at all.

The games we will be talking about though today are what are known as combinatorial games. In these games there is no change, or randomness (i.e. dice, etc.) and we always see what our opponent is doing. There is also no diplomacy, you are always trying to beat your opponent. Chess and checkers are games of this nature as well as Go and Mancala and Blockus. These games are studied by mathematicians because we can program computers to play them and thus develop strategies for playing them using mathematics. We do this by creating what are called game trees. Lets play a game first so we can see how a game tree is generated.

**The GAME!**
We will now play a very simple game that everyone here probably knows, Tic-tac-toe.

Have the students break up into pairs to play. Pass out pencils and paper or ask them to get some out. Have the students play for awhile. Say roughly 10 minutes or so (have them rotate around and switch partners), but not until they have exhausted all strategies. Have them stop and ask them the following questions:

• How many turns does each player have in one game?
• Who do you think has a better chance of winning the first person or the second and why?
• How many different positions or places does each player have on each turn? i.e. so if I was starting out how many places can I place my mark before starting, after I take my turn, how many places does the second person have after she takes her turn and so on and so on.
• Is there any way that if neither of you make a “mistake” that someone can win?
• Do you like this game? Why or why not?
• What are some things that you would change about this game?
• How could you and your partner change the rules, the board, the “pieces”, etc around to make it harder or more fun or more complicated?

Now show the game tree for 1/3 of the Tic Tac Toe game attached. Show the students how this tree could be used to tell a computer how to play a human by showing all the combinations of play for the game and derive the combinations on the board.
If we ignore symmetries of the board there are: 131,184 finished games are won by (X); 77,904 finished games are won by (O); 46,080 finished games are drawn.

With symmetries: 91 unique positions are won by (X); 44 unique positions are won by (O); 3 unique positions are drawn.

Compare this with the upper limit of possible games in Go, which is $10^{10^{171}}$. This is far more than the number of atoms in the universe, which is around $10^{82}$. This is why it is practically impossible to program a computer to play Go well (experienced little kids can beat the most advanced computers). This is not the case for Chess, however, where super computers can beat and tie and lose to even the best players (i.e., they are fairly on par).

Now have the students each draw a series of four lines up and down and four lines right and left all crossing each other. Now we will draw a large box around them. We will have a grid that looks like this:

We will now again play tic-tac-toe but we will have to get 4 in a row this time.

- What is different about this version?
- What is the same?
- Does anyone have an advantage?
- What are some strategies that you or your partner used to win?
- When can you tell someone has won or lost?

These types of games are known as \((m,n,k)\) games, where the \(m,n\) are natural numbers that dictated the board or matrix size while the \(k\) denotes the number needed to get in a row to win. Therefore Tic Tac Toe is a 3,3,3-game while Gomuku is (technically) a 19,19,5-game. What if we played a 1,5,2-game? What can you say about who wins and why? Can we use this logic to say anything about a certain class of games? Have the students experiment with other variations of size and "in a row" to determine why certain boards are "solved" and others not.

NOW if they have not already we will play Gomoku (19,19,5-game).
Gomoku Game Board

Player 1:________________
Player 2:________________
Reflection:
Over all I think this lesson/activity has gone over very well. I have used it now with a range of students from middle school to high school and adults and everyone seems to enjoy it. I have used it mostly in Math Circle but twice in my classroom with both Ann and Robert's class.

Some notes in delivery of the lesson:
This was part of a larger curriculum around game theory, which included games such as Puppies and Kittens, a version of nim with two piles as well as a prisoner dilemma game and a dollar auction. The students were primed to be looking for different strategies in which to create a winning position (or minimal lose).

In addition to the games matrices listed above, some others to help in the students determining how the mathematics works is to have them play, (n,1,2), (n,1,3), (n,2,k), (n,3,k), and finally (n,n,k). These game boards give an intuition in degrees of freedom for movement and seeing how general strategies play out for "base" games. This lesson could be used to teach induction as well. In that we could have the students begin with basic game states and then start to build up, as suggested above, to form a game winning strategy (if one exists).